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INVOLUTORY UNITARY MATRICES OF INTEGERS ASSOCIATED WITH CERTAIN GEOMETRIC TRANSFORMATIONS

By T. G. ROOM (*Sydney*)

[Received 16 June 1948]

In this paper three systems, two geometric and one algebraic, are discussed which depend on a certain simple 'groupoid', of type G_3, in the sense in which this term is used by Richardson (1942). This groupoid has an arbitrary number of generators, with the properties: (i) the operators are fully associative, (ii) every product of an odd number of operators is an operator, (iii) for every operator, $\mathscr{A} = \mathscr{A}^{-1}$, and (iv) for every set of three operators, $\mathscr{A}\mathscr{B}\mathscr{C} = \mathscr{C}\mathscr{B}\mathscr{A}$.

In Part I the operators are the involutory self-transformations of a plane cubic curve determined by the pencils of lines drawn through points of the curve. In Part II they are the involutory self-transformations of a surface which has on it certain rational curves and a line of multiplicity three less than the order of the surface. In Part III they are certain matrices of integers which satisfy: (i) $\mathbf{Y}^2 = \mathbf{I}$, and (ii) $\mathbf{Y\Gamma Y}' = \mathbf{\Gamma}$, i.e., with (i), $\mathbf{Y\Gamma} = (\mathbf{Y\Gamma})'$, where $\mathbf{\Gamma}$ is a given symmetrical matrix of integers in which the leading element is zero, the remaining elements of the leading diagonal are equal, the remaining elements of the first row and column are all 1, and all the other elements are unrestricted. The matrix is set out formally in Theorem [7.4] and equation (9.1).

Between I and II there is a simple geometrical link. The connexion between II and III rests on the theorem: *if curves* $c_1, \ldots, c_r$, *or* $\mathbf{c}$, *with intersection matrix* $(\{c_i, c_j\}) = \mathbf{\Gamma}$, *form a base for algebraic curves on a surface which has a birational self-transformation* $\mathscr{T}$ (*in which the exceptional elements are of a sufficiently restricted type*), *and the transforms* $\mathscr{T}\mathbf{c}$ *of the curves* $\mathbf{c}$, *when expressed in terms of these curves, are given by* $(\mathscr{T}\mathbf{c}) = \mathbf{Tc}$, *then* $\mathbf{T\Gamma T}' = \mathbf{\Gamma}$.†

† See Room (1948), Theorem 1.1. This paper will be referred to as M.I. I wish to thank Mr. L. Roth of the Royal College of Science, London, for having pointed out to me that several of the results in it had been anticipated by Severi (1906), (1911), Enriques (1906), (1912), and Snyder and Sharpe (1914), (1915).

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I. Systems of Involutory Transformations of a Plane Cubic Curve

1. The principal relation among the transformations

Let u be a fixed general plane cubic curve. I shall describe three collinear points of the curve as being *in syzygy*. Lines through any point a of u cut out on u an involution of pairs of points in syzygy with a, i.e. they determine a self-transformation of u, which I shall denote by the corresponding script capital $\mathscr{A}$ or by $[a]$. If three points are in syzygy each determines an involutory transformation in which the other two points correspond, i.e.

[1.1] *If a, b, c are in syzygy on u, then*

$$\mathscr{A}b = c, \quad \mathscr{A}c = b, \quad \mathscr{B}c = a, \quad \mathscr{B}a = c, \quad \mathscr{C}a = b, \quad \mathscr{C}b = a$$

(or, in the alternative notation, $[a]b = c$, etc.).

The fundamental relation among the transformations determined by points of u is

[1.2] *If p, q, r are any three points of u, then*

$$\mathscr{R} = \mathscr{P}2\mathscr{R}\mathscr{P}2.$$

To prove this, take any point h of u and construct the following sets of three points in syzygy: qhk_1, k_1pk_2, k_2rk_3, k_3qk_4, k_4ph^*. The first, third, and fifth of these sets contain three distinct sets of three collinear points, that is, nine associated points, and these nine points may be arranged as the second and fourth sets, together with hrh^*. Thus

$$\mathscr{R}h = h^*,$$

and, from the sequence of syzygies,

$$2h = k_1, \quad \mathscr{P}k_1 = k_2, \quad \mathscr{R}k_2 = k_3, \quad 2k_3 = k_4, \quad \mathscr{P}k_4 = h^*.$$

This proves the theorem.

Now let q, r, s be in syzygy, so that $2r = s$; then by much the same method as above we may prove

$$[\mathscr{P}s] = 2\mathscr{P}\mathscr{R}.$$

This provides a second and more useful form of the principal theorem, namely:

[1.3] *If q, r, s are three points in syzygy, and p is any other point, then $2\mathscr{P}\mathscr{R}$ is the involutory transformation of u determined by the point $\mathscr{P}s$.*

In terms of p, q, r alone we may write this without ambiguity in the form

$$[\mathscr{P}2r] = [\mathscr{P}\mathscr{R}q] = 2\mathscr{P}\mathscr{R} = \mathscr{R}\mathscr{P}2,$$

or, in the alternative notation,

$$[[p][q]r] = [q][p][r], \text{ etc.}$$

As a consequence of the relation $2\mathscr{P}\mathscr{R} = \mathscr{R}\mathscr{P}2$ we have:

[1.4] *The transformation corresponding to the product of any number of factors $\mathscr{P}_1, \mathscr{P}_2, \ldots$ is unchanged if the order of any odd number of consecutive factors is reversed.*

Now suppose that elliptic parameters ϕ are assigned to points of u in such a way that collinear points have parameters whose sum is a period, i.e. if q, r, s are in syzygy, then

$$\phi_q + \phi_r + \phi_s \equiv 0.$$

The equations representing the transformations then take the form

$$\phi_{2r} \equiv -\phi_q - \phi_r,$$

and the principal theorem may be written as

[1.5] *If ϕ_p, ϕ_q, ϕ_r are the elliptic parameters of any three points of u, then*

$$\phi_{2\mathscr{P}\mathscr{R}} = \phi_{\mathscr{R}\mathscr{P}2} \equiv \phi_q - \phi_p + \phi_r,$$

where $\phi_{2\mathscr{P}\mathscr{R}}$ is written for the parameter of the point which determines the transformation $2\mathscr{P}\mathscr{R}$.

2. The one-fold syzygetic system of points and transformations

The point $\mathscr{P}p$ corresponding to p itself in the transformation $\mathscr{P}$ is the third point of incidence of the tangent at p with u.† This point determines a transformation $[\mathscr{P}p]$, or $[[p]p]$, and thence further points $[\mathscr{P}p]p$, $[\mathscr{P}p]\mathscr{P}p$, and so on. First, since one of the tangents from $\mathscr{P}p$ to u touches u at p, p is self-corresponding in $[\mathscr{P}p]$, i.e.

[2.1] $[\mathscr{P}p]p = p$.

Since p, p, $\mathscr{P}p$ are three points in syzygy, we may replace the points q, r, s in Theorem [1.3] by them in any order. Write then successively for p, q, r, s: (i) $\mathscr{P}p, p, p, \mathscr{P}p$, (ii) $h, p, \mathscr{P}p, p$, and (iii) $h, p, p, \mathscr{P}p$; we obtain:

[2.2] (i) $[[\mathscr{P}p]\mathscr{P}p] = \mathscr{P}[\mathscr{P}p]\mathscr{P}$,

(ii) $[\mathscr{H}p] = \mathscr{P}\mathscr{H}[\mathscr{P}p] = [\mathscr{P}p]\mathscr{H}\mathscr{P}$,

(iii) $[\mathscr{H}\mathscr{P}p] = \mathscr{P}\mathscr{H}\mathscr{P}$.

Setting out from p, construct the system of points on u which are

† We assume that p is not an inflexion.

either on the tangent at a point already constructed, or in syzygy with two points already constructed. This system of points, and the system of transformations corresponding to them, will be called a *one-fold syzygetic system*, $\mathscr{S}^1(p)$. By induction from Theorem [2.2], treating 'p', and 'h' as points already constructed, we find:

[2.3] *Every transformation of the one-fold syzygetic system $\mathscr{S}^1(p)$ may be expressed as the product of an odd number of factors $\mathscr{P}$, $[\mathscr{P}p]$.*

That is, every transformation may be expressed as $\mathscr{P}([\mathscr{P}p]\mathscr{P})^\alpha$, where α is an integer, positive, negative, or zero.

If ϕ_p is the elliptic parameter of p, we have

$$\phi_{\mathscr{P}p} = -2\phi_p, \qquad \phi_{[\mathscr{P}p]\mathscr{P}p} = 4\phi_p, \quad \text{etc.}$$

We may summarize the results thus:

[2.4] *The system $\mathscr{S}^1(p)$ consists of all the points p_α of elliptic parameters $(3\alpha+1)\phi_p$.† The transformation $\mathscr{P}_\alpha$ determined by p_α is expressible as the product $\mathscr{P}([\mathscr{P}p]\mathscr{P})^\alpha$, i.e. $\mathscr{P}_0(\mathscr{P}_{-1}\mathscr{P}_0)^\alpha$.*

Although we do not need them in the subsequent development (except in the discussion at the end of § 6), it is of some interest to inquire what is the part played by points of which the parameters are other special multiples of ϕ_p.

(i) *Other integer multiples.* Let j be the point of parameter zero, i.e. the inflexion at the origin of the particular system of elliptic parameters selected. Given only u and the point p on it, there are then, nine points j on u. Each system $\mathscr{S}^1(\mathscr{J}p)$ consists of points $(\mathscr{J}p)_\beta$ of parameters $(-3\beta-1)\phi_p$. The points p_β, j, and $(\mathscr{J}p)_\beta$ are in syzygy, i.e. $\mathscr{J}(p_\beta) = (\mathscr{J}p)_\beta$. Further, if j_γ is the point of parameter $3\gamma\phi_p$, then $\mathscr{P}_\alpha(\mathscr{J}p)_{\alpha+\gamma} = j_\gamma$, for all α. Thus:

[2.5] *Associated with a given point p and a given inflexion j, there are two syzygetic systems $\mathscr{S}^1(p)$ and $\mathscr{S}^1(\mathscr{J}p)$ and a system $\mathscr{W}^1(p)$ containing points of parameters $(3\alpha+1)\phi_p$, $(-3\beta-1)\phi_p$, and $3\gamma\phi_p$ respectively. Any two points taken from different systems are in syzygy with a point of the third system.*

(ii) *The system $\Sigma^1(p)$.* From p draw the four tangents to u, with points of contact, say, t_{-1}, of parameters $-\frac{1}{2}(\phi_p+\text{period})$. Then from each of these draw again the four tangents, obtaining the sixteen points t_{-2} of parameters $\frac{1}{4}(\phi_p+\text{period})$. From these and p

† The groupoid is finite if, for some α, $(3\alpha+1)\phi_p \equiv \phi_p$. We assume that this condition is not satisfied.

we obtain sixteen points $-\frac{5}{4}(\phi_p+\text{period})$, and from these and t_{-1} sixteen points $\frac{7}{4}(\phi_p+\text{period})$. The tangent at any of these last gives one of the four points $-\frac{7}{2}(\phi_p+\text{period})$, and the tangents at these meet in the point $7\phi_p$, namely p_2. Continuing in this way we obtain:

[2.6] *The system $\Sigma^1(p)$ consists of all the points of parameters $(-2)^{-\mu}(3\alpha+1)(\phi_p+period)$. It depends on p alone, and not the choice of a particular inflexion as origin of the elliptic coordinates.*

3. The two-fold syzygetic system

From three points a, b, c in syzygy on u we build a two-fold syzygetic system $\mathscr{S}^2(a,b)$ of points and transformations, by constructing the third point of u on the tangent to u at any point already constructed and the point in syzygy with two points already constructed. The principal result to be proved is that the transformation corresponding to any point of this system may be expressed as the product of an odd number of factors $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, and we find a standard form for such products.

First consider products in which $\mathscr{C}$ precedes $\mathscr{B}$; there are three types, and, using Theorem [1.4], we may rearrange each as a product in which $\mathscr{C}$ follows $\mathscr{B}$: thus (i) $\mathscr{C}\mathscr{A}\mathscr{B} = \mathscr{B}\mathscr{A}\mathscr{C}$, (ii) $\mathscr{A}\mathscr{C}\mathscr{B} = \mathscr{B}\mathscr{C}\mathscr{A}$, and (iii) $\mathscr{B}\mathscr{C}\mathscr{B} = \mathscr{B}\mathscr{A}\mathscr{A}\mathscr{C}\mathscr{B} = \mathscr{B}\mathscr{A}\mathscr{B}\mathscr{C}\mathscr{A}$. Thus after a finite succession of these rearrangements (with the insertion and removal of identity factors), we find

[3.1] *The product of any number of factors $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ may be reduced to one of the two forms $(\mathscr{A}\mathscr{B})^\beta(\mathscr{A}\mathscr{C})^\gamma$, or $\mathscr{A}(\mathscr{A}\mathscr{B})^\beta(\mathscr{A}\mathscr{C})^\gamma$, where β, γ are integers.*

By another succession of rearrangements it may be proved that

$$\mathscr{A}(\mathscr{A}\mathscr{B})^\beta(\mathscr{A}\mathscr{C})^\gamma = (\mathscr{C}\mathscr{A})^\gamma(\mathscr{B}\mathscr{A})^\beta\mathscr{A} = \{\mathscr{A}(\mathscr{A}\mathscr{B})^\beta(\mathscr{A}\mathscr{C})^\gamma\}^{-1},$$

and thus, since the parity of the number of terms is unchanged by the rearrangements,

[3.2] *The product of any odd number of factors $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ is an involutory transformation.*

Since, from Theorem [1.3], $[\mathscr{H}a] = \mathscr{B}\mathscr{H}\mathscr{C}$, if $\mathscr{H}$ belongs to $\mathscr{S}^2(a,b)$ and is expressible as a product, then $[\mathscr{H}a]$, which also belongs to $\mathscr{S}^2(a,b)$, is also expressible as a product; whence an inductive argument shows that any member of $\mathscr{S}^2(a,b)$ may be so expressed. Now consider

$$\mathscr{F} = \mathscr{A}(\mathscr{A}\mathscr{B})^\beta(\mathscr{A}\mathscr{C})^\gamma,$$

and suppose that $\beta > 0$, $\beta-\gamma = 2\mu > 0$ (other cases may be dealt with similarly). Then

$$\begin{aligned}\mathscr{F} &= (\mathscr{B}\mathscr{A})^{\gamma}(\mathscr{B}\mathscr{A})^{2\mu-1}\mathscr{B}(\mathscr{A}\mathscr{C})^{\gamma}\\ &= (\mathscr{B}\mathscr{A})^{\gamma}\mathscr{B}\mathscr{A}\ldots\mathscr{A}\mathscr{B}\mathscr{A}\mathscr{B}\mathscr{A}\ldots\mathscr{A}\mathscr{B}(\mathscr{A}\mathscr{C})^{\gamma}\\ &= (\mathscr{B}\mathscr{A})^{\gamma}[[\ldots[[[\mathscr{A}\mathscr{B}b]\mathscr{A}a]\mathscr{B}b]\ldots\mathscr{A}a]\mathscr{B}b](\mathscr{A}\mathscr{C})^{\gamma}\\ &\qquad\text{(working outwards from the middle and using Theorem [2.2] (iii))}\\ &= [[\ldots[[[\ldots[[[\mathscr{A}\mathscr{B}b]\mathscr{A}a]\mathscr{B}b]\ldots\mathscr{A}a]\mathscr{B}b]\mathscr{A}a]a]\ldots\mathscr{A}a]a]\\ &\qquad\text{(taking alternately } \mathscr{A}(\ldots)\mathscr{A} \text{ and } \mathscr{B}(\ldots)\mathscr{C}),\end{aligned}$$

there being μ terms $\mathscr{B}b$, $\gamma+\mu-1$ terms $\mathscr{A}a$, and γ terms a. Thus

[3.3] *Every transformation of the two-fold system $\mathscr{S}^2(a,b)$ may be expressed in a unique way as a product $\mathscr{A}(\mathscr{A}\mathscr{B})^{\beta}(\mathscr{A}\mathscr{C})^{\gamma}$, and every product of an odd number of factors $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ belongs to the system $\mathscr{S}^2(a,b)$.*

Take the elliptic parameters of a, b, c to be ϕ_a, ϕ_b, ϕ_c, with

$$\phi_a+\phi_b+\phi_c \equiv 0.$$

From Theorem [1.5] we have

$$\phi_{\mathscr{B}\mathscr{A}\mathscr{B}} \equiv -\phi_a+2\phi_b,$$

$$\phi_{\mathscr{B}\mathscr{A}\mathscr{B}\mathscr{A}\mathscr{B}} \equiv -\phi_a+2\phi_b-\phi_a+\phi_b, \text{ etc.},$$

and thus, if $\mathscr{F} = \mathscr{B}(\mathscr{A}\mathscr{B})^{\beta-1}(\mathscr{A}\mathscr{C})^{\gamma}$,

$$\phi_{\mathscr{F}} \equiv -(\beta+\gamma-1)\phi_a+\beta\phi_b+\gamma\phi_c,$$

That is

[3.4] *The involutory transformation $\mathscr{A}(\mathscr{A}\mathscr{B})^{\beta}(\mathscr{A}\mathscr{C})^{\gamma}$ is determined by the point with elliptic parameter $\alpha\phi_a+\beta\phi_b+\gamma\phi_c$, where $\alpha+\beta+\gamma = 1$.* This is the more symmetrical form, but we may state the result in more general terms as:

[3.5] *The syzygetic system $\mathscr{S}^2(a,b)$ consists of the points with elliptic parameters $\lambda\phi_a-(\lambda+3\mu-1)\phi_b$, and the corresponding transformations.*

4. The syzygetic system determined by r points

Let a_i $(i = 1,\ldots,r)$ be r general points of the plane cubic curve u.† Denote by $\mathscr{A}_i$ the transformation determined by a_i and write

$$a_{ij} = a_{ji} = \mathscr{A}_i a_j = \mathscr{A}_j a_i, \qquad \mathscr{A}_{ij} = \mathscr{A}_{ji} = [\mathscr{A}_i a_j] = [a_{ij}],$$

and

$$a_{ii} = \mathscr{A}_i a_i, \qquad \mathscr{A}_{ii} = [\mathscr{A}_i a_i].$$

† By 'general' in this sense it is meant that the syzygetic system determined as below by any $r-1$ points does not contain the remaining point, and that none of the points is an inflexion.

In this notation Theorem [1.2] becomes

[4.1] $\mathscr{A}_k = \mathscr{A}_i \mathscr{A}_j \mathscr{A}_k \mathscr{A}_i \mathscr{A}_j$,

while [1.4] and [2.2] become

[4.2] (i) $\mathscr{A}_{ij} = [\mathscr{A}_j a_i] = \mathscr{A}_{ik} \mathscr{A}_j \mathscr{A}_k = \mathscr{A}_{ii} \mathscr{A}_j \mathscr{A}_i$;

(ii) $[\mathscr{A}_j a_{ii}] = \mathscr{A}_i \mathscr{A}_j \mathscr{A}_i$;

(iii) $[\mathscr{A}_{ii} a_{ii}] = \mathscr{A}_i [\mathscr{A}_i a_i] \mathscr{A}_i = \mathscr{A}_i \mathscr{A}_{ii} \mathscr{A}_i$.

From Theorem [4.2] (i) it follows that all transformations $\mathscr{A}_{ii}$, $\mathscr{A}_{ij}$ may be expressed as products of factors $\mathscr{A}_i$ together with one other factor, say $\mathscr{A}_{11}$. For example

$$\mathscr{A}_{23} = \mathscr{A}_1 \mathscr{A}_2 \mathscr{A}_1 \mathscr{A}_3 \mathscr{A}_{11}.$$

Consider now how the r-fold syzygetic system $\mathscr{S}^r(a_i)$ can be built up. We have

$$[\mathscr{A}_k a_{ij}] = [\mathscr{A}_{ij} a_k] = \mathscr{A}_{kp} \mathscr{A}_{ij} \mathscr{A}_p, \quad \text{from [4.2] (i)},$$

whence

$$[\mathscr{A}_{ij} a_{kl}] = [\mathscr{A}_{kl} a_{ij}] = [\mathscr{A}_{kl} a_p] \mathscr{A}_{ij} \mathscr{A}_p = \mathscr{A}_{pq} \mathscr{A}_{kl} \mathscr{A}_q \mathscr{A}_{ij} \mathscr{A}_p,$$

and $$[[\mathscr{A}_{ij} a_k] a_l] = \mathscr{A}_{lp} [\mathscr{A}_{ij} a_k] \mathscr{A}_p = \mathscr{A}_{lp} \mathscr{A}_{kq} \mathscr{A}_{ij} \mathscr{A}_q \mathscr{A}_p,$$

where the subscripts need not all be distinct. Proceeding in this way, and using an argument similar to that used in proving Theorem [3.3], we may prove that

[4.3] *All transformations of the syzygetic system $\mathscr{S}^r(a_i)$ may be expressed as the products of odd numbers of factors from the set of $r+1$ factors $\mathscr{A}_1, \ldots, \mathscr{A}_r$, and $\mathscr{A}_{11}$. Any product of an odd number of transformations of the system $\mathscr{S}^r(a_i)$ is a member of the system.*

As in Theorem [3.3] the product may be reduced to a standard form, say $\mathscr{A}_{11}(\mathscr{A}_1 \mathscr{A}_{11})^{\alpha_1}(\mathscr{A}_2 \mathscr{A}_{11})^{\alpha_2} \ldots (\mathscr{A}_r \mathscr{A}_{11})^{\alpha_r}$, where $\alpha_1, \ldots, \alpha_r$ are integers: e.g.

$$\mathscr{A}_{23} = \mathscr{A}_{11}(\mathscr{A}_1 \mathscr{A}_{11})^{-2}(\mathscr{A}_2 \mathscr{A}_{11})(\mathscr{A}_3 \mathscr{A}_{11}).$$

For most purposes, this notation, using one or two subscripts, is most convenient, but certain results are expressible more compactly in terms of a notation based on the elliptic parameters of the points. Let the elliptic parameter of a_i be ϕ_i, and suppose that $\sum \lambda_i \phi_i$ is a point of $\mathscr{S}^r(a_i)$. I shall denote this point by $a_{\lambda_1 \ldots \lambda_r}$ (there is no confusion with the other notation, if, as we assume, $r > 2$), or by $a_{\boldsymbol{\lambda}}$, and denote the corresponding transformation by $\mathscr{A}_{\lambda_1 \ldots \lambda_r}$, or $\mathscr{A}_{\boldsymbol{\lambda}}$, or $[a_{\boldsymbol{\lambda}}]$. I shall use the sign '+' as for ordinary addition of vectors, so that the point $a_{\boldsymbol{\lambda}+\boldsymbol{\mu}}$ is that with elliptic parameter $\sum (\lambda_i + \mu_i)\phi_i$.

The connexion between the syzygies and the transformations (Theorem [1.1]) is expressed by

[4.4] $\mathscr{A}_{\boldsymbol{\lambda}} a_{\boldsymbol{\mu}} = a_{\boldsymbol{\nu}}$, *where* $\boldsymbol{\lambda}+\boldsymbol{\mu}+\boldsymbol{\nu} = \mathbf{0}$.

Using this relation, and an induction based on $\mathscr{A}_1 = \mathscr{A}_{10000\ldots}$, $\mathscr{A}_2 = \mathscr{A}_{01000\ldots}$, $\mathscr{A}_{12} = \mathscr{A}_{-1-1000\ldots}$, we find

[4.5] *A symbol* $a_{\boldsymbol{\lambda}}$ *can represent a point of the system* $\mathscr{S}^r(a_i)$ *only if* $\sum \lambda_i = 3\lambda+1$, *where* λ_i, λ *are integers.*

Theorem [1.5] takes the form

[4.6] (i) $\mathscr{A}_{\boldsymbol{\lambda}} \mathscr{A}_{\boldsymbol{\mu}} \mathscr{A}_{\boldsymbol{\nu}} = \mathscr{A}_{\boldsymbol{\lambda}-\boldsymbol{\mu}+\boldsymbol{\nu}}$;†

(ii) $\mathscr{A}_{\boldsymbol{\rho}_0} \mathscr{A}_{\boldsymbol{\sigma}_1} \mathscr{A}_{\boldsymbol{\rho}_1} \mathscr{A}_{\boldsymbol{\sigma}_2} \mathscr{A}_{\boldsymbol{\rho}_2} \ldots \mathscr{A}_{\boldsymbol{\sigma}_s} \mathscr{A}_{\boldsymbol{\rho}_s} = \mathscr{A}_{\boldsymbol{\tau}}$, where $\boldsymbol{\tau} = \sum_0^s \boldsymbol{\rho}_i - \sum_1^s \boldsymbol{\sigma}_i$.

In consequence of Theorem [4.5] the systems $\mathscr{S}^r$ separate into three classes according to the residue of r modulo 3:

(i) $r = 3s$: there is no point symmetrically related to all the points a_i;

(ii) $r = 3s+1$: there is a one-fold syzygetic system $\mathscr{S}^1(a_{1111\ldots})$ of points each symmetrically related to all the points a_i;

(iii) $r = 3s-1$: there is a one-fold syzygetic system $\mathscr{S}^1(a_{-1-1-1\ldots})$ of points each symmetrically related to all the points a_i. In particular $a_{-1-1-1\ldots} = a_*$, say, and $a_1, \ldots, a_r$ form a symmetrical set of $3s$ points (the section of u by a curve of order s). It might be expected that it would be possible to use products of the $r+1$ transformations $\mathscr{A}_*, \mathscr{A}_1, \ldots, \mathscr{A}_r$ to determine the whole system $\mathscr{S}^r(a_i)$, but this is not so except when $s = 1$ (the case dealt with in § 3). Take, for instance, $s = 2$, then from [4.6] (ii),

$$\begin{aligned}\mathscr{A}_* &= \mathscr{A}_{-1-1-1-1-1} \\ &= \mathscr{A}_{11} \mathscr{A}_2 \mathscr{A}_1 \mathscr{A}_3 \mathscr{A}_1 \mathscr{A}_4 \mathscr{A}_1 \mathscr{A}_5 \mathscr{A}_{11}.\end{aligned}$$

For a general value of s we shall be able to express $\mathscr{A}_*$ as a product involving $\mathscr{A}_1, \ldots, \mathscr{A}_r$, and s terms $\mathscr{A}_{11}$, and thus shall not be able to obtain $\mathscr{A}_{11}$ as a product of $\mathscr{A}_*, \mathscr{A}_1, \ldots, \mathscr{A}_r$.

II. Some Surfaces with Involutory Self-transformations

5. $\mathscr{R}$-curves, and $\mathscr{R}$-transformations

Let π be a surface of order $N+2$ in space of three dimensions, specialized only in such a way that it has on it an $(N-1)$-ple line v,

† Or $\mathscr{A}_{\boldsymbol{\rho}} \mathscr{A}_{\boldsymbol{\sigma}} = \mathscr{A}_{\boldsymbol{\rho}'} \mathscr{A}_{\boldsymbol{\sigma}'}$ where $\boldsymbol{\rho}-\boldsymbol{\sigma} = \boldsymbol{\rho}'-\boldsymbol{\sigma}'$, but $\mathscr{A}_{\boldsymbol{\rho}} \mathscr{A}_{\boldsymbol{\sigma}}$, although a transformation of u, is not involutory, i.e. not a member of the 'groupoid'.

and an irreducible rational curve a of order m to which v is $(m-1)$-secant, i.e. a is an irreducible curve such that any plane through v has only one variable intersection with it. Since a single line may be drawn through any point of space to meet v and a in distinct points, and this line meets π in just two further points, it follows that:

[5.1] *There is an involutory self-transformation $\mathscr{A}$ of π in which the join of any pair of corresponding points meets the line v and the curve a.*

In relation to a given surface π and line v of multiplicity three less than the order of π, the curve a will be referred to as an *$\mathscr{R}$-curve*, and the corresponding transformation $\mathscr{A}$ as an *$\mathscr{R}$-transformation*. In the work which follows we suppose the surface and multiple line are fixed, and the $\mathscr{R}$-curves are assigned as required on the surface.

Any plane ϖ through v cuts π in a cubic curve u which meets a in one point, a_u say. If h is any point of u, and h^* is the point into which it transforms, the line hh^* meets v and therefore lies in ϖ; i.e. h^* also lies on u, and a, h, h^* are three points in syzygy on the curve. Thus

[5.2] *The $\mathscr{R}$-transformation $\mathscr{A}$ determined by an $\mathscr{R}$-curve, a, determines an involutory self-transformation $\mathscr{A}_u$ of each plane cubic curve of the pencil coplanar with v.*

Now let b be a second $\mathscr{R}$-curve, and b_u the point in which it meets u. Then $\mathscr{A}b_u$ also lies on u, and is the only point of the transformed curve $\mathscr{A}b$ which does so. Thus $\mathscr{A}b$ is a curve, irreducible if b is, which meets each u in only one point, and is therefore an $\mathscr{R}$-curve; i.e.

[5.3] *The $\mathscr{R}$-transform of any $\mathscr{R}$-curve is an $\mathscr{R}$-curve.*

In particular $\mathscr{A}a$, the curve cut out by the ruled surface of lines which meet v and are tangents to π at points of a, is an $\mathscr{R}$-curve.

The three $\mathscr{R}$-curves a, b, and $c = \mathscr{A}b$, together with v, form the complete intersection of a ruled surface with π: that is, each is the transform of either of the others in the transformation determined by the third—precisely the relation expressed symbolically in Theorem [1.1]. We are justified in describing such a set of three curves as being *in syzygy*; they have the property of meeting each curve u in points in syzygy. It follows further that, abstractly, the system of $\mathscr{R}$-transformations determined by an assigned set of $\mathscr{R}$-curves on a surface π with a given multiple line v is identical with the system of transformations determined by the set of points in which the curves meet any one of the curves u, and consequently:

[5.4] *A set of r $\mathscr{R}$-curves $a_1, \ldots, a_r$ determines an r-fold syzygetic system of $\mathscr{R}$-curves and $\mathscr{R}$-transformations $\mathscr{R}^r(a_i)$. All the relations connecting points and transformations of a system $\mathscr{S}^r(a_i)$ on a cubic curve which are set out in § 3 may be interpreted as relations connecting the $\mathscr{R}$-curves and $\mathscr{R}$-transformations of the system of $\mathscr{R}^r(a_i)$ on the surface.*

III. Unitary Matrices associated with the $\mathscr{R}$-transformations

6. Matrices associated with a one-fold syzygetic system

An $\mathscr{R}$-transformation $\mathscr{A}$ of π is one in which every point transforms into a single point (the $N-1$ 'places' on π at a point of v being counted separately). If then we have a base for algebraic curves on π, say c_i, or, collectively, $\mathbf{c}$, and $\mathscr{A}$ is such that the transform $\mathscr{A}(\sum \alpha_i c_i)$, or $\mathscr{A}(\boldsymbol{\alpha}'\mathbf{c})$, of any curve $\boldsymbol{\alpha}'\mathbf{c}$ is given by

$$\mathscr{A}(\boldsymbol{\alpha}'\mathbf{c}) = \boldsymbol{\alpha}'\mathbf{A}\mathbf{c},$$

where $\mathbf{A}$ is a constant matrix of integers (depending, that is, only on π, the transformation, and the base, and not on the particular set of values α_i), we may apply the results of M.I.† Thus, if the number of points common to two curves p, q is written $\{p, q\}$, and

$$(\{c_i, c_j\}) = (\{c_j, c_i\}) = \boldsymbol{\Gamma} = \boldsymbol{\Gamma}',$$

then
$$\mathbf{A}\boldsymbol{\Gamma}\mathbf{A}' = \boldsymbol{\Gamma}. \tag{6.1}$$

Also, since $\mathscr{A}$ is involutory,

(inv-$\mathbf{A}$)
$$\mathbf{A}^2 = \mathbf{I}, \tag{6.2}$$

and therefore

(sym-$\mathbf{A}\boldsymbol{\Gamma}$)
$$\mathbf{A}\boldsymbol{\Gamma} = (\mathbf{A}\boldsymbol{\Gamma})'. \tag{6.3}$$

I shall describe conditions (6.2) and (6.3) as 'inv-$\mathbf{A}$' and 'sym-$\mathbf{A}\boldsymbol{\Gamma}$'.

Suppose now that π has a single assigned $\mathscr{R}$-curve, p.‡ The curves with which we are concerned are equivalent to the sums of integer

† It should be noted that, if $\mathscr{P}$, $\mathscr{Q}$ are two transformations, and $\mathbf{P}$, $\mathbf{Q}$ the corresponding matrices on the same base, and

$$\boldsymbol{\beta}'\mathbf{c} = \mathscr{P}\mathscr{Q}(\boldsymbol{\alpha}'\mathbf{c}),$$

then
$$\boldsymbol{\beta}' = \boldsymbol{\alpha}'\mathbf{Q}\mathbf{P};$$

i.e. $\mathbf{Q}\mathbf{P}$ is the matrix corresponding to the transformation $\mathscr{P}\mathscr{Q}$. If the transformations concerned are $\mathscr{R}$-transformations, and we have an odd number of factors, then, in consequence of Theorem [1.4], no reversal need be made.

‡ Cf. M.I. § 9. The quartic surface is given by $N = 2$. For comparison, the base used in M.I., in the notation of this paper, consists of the three curves $u+v$, v, p, the first being a complete plane section.

multiples of v the multiple line, u a plane cubic residual to it, and the rational curve p.† I write the curves always in the order u, v, p, and shall refer to them as c_{-1}, c_0, c_1.

Let h be a curve on π of order $\nu_{-1}+\nu_0$ which meets c_i in ν_i points. By degeneration methods (cf. M.I. 41, footnote) we may prove that transversals from points of h to v and p form a surface of order

$$(m+1)\nu_{-1}+\nu_0-\nu_1,$$

on which the multiplicity of v is

$$m\nu_{-1}+\nu_0-\nu_1$$

and the multiplicity of p is ν_{-1},

so that the order of $\mathscr{P}h$, which is the residual intersection of this surface with π, is $(N+2m+1)\nu_{-1}+2\nu_0-3\nu_1$. The curves h and $\mathscr{P}h$ meet u in the same number ν_{-1} of points, so that $\mathscr{P}h$ meets v in $(N+2m)\nu_{-1}+2\nu_0-3\nu_1$ points. Suppose now that

$$h = \boldsymbol{\alpha}'\mathbf{c}, \qquad \mathscr{P}h = \boldsymbol{\alpha}'\mathbf{P}_v\mathbf{c}.$$

Then [M.I., equation (1.9)], if $\mathscr{P}h$ meets c_i in $\mathscr{P}\nu_i$ points,

$$\mathscr{P}\boldsymbol{\nu} = \mathbf{P}_v^{-1}\boldsymbol{\nu} = \mathbf{P}_v\boldsymbol{\nu}.$$

We have $\mathscr{P}u = u$, and, from the work above,

$$\mathscr{P}\nu_0 = (N+2m)\nu_{-1}+2\nu_0-3\nu_1.$$

Thus the first two rows of $\mathbf{P}_v$ are $(1, 0, 0)$ and $(N+2m, 2, -3)$. The third row may be determined from inv-$\mathbf{P}_v$. We find

[6.1] *With base u, v, p,*

$$\mathbf{P}_v = \begin{pmatrix} 1 & 0 & 0 \\ N+2m & 2 & -3 \\ N+2m & 1 & -2 \end{pmatrix}.$$

† We do not need to assume that u, v, p actually form a base; what we require is that $\mathscr{P}u$, $\mathscr{P}v$, $\mathscr{P}p$ are all expressible in terms of u, v, p, and we do in fact find the actual expressions. The position here is different from that in M.I., where, to some extent, the investigation was concerned with finding all the curves on the surface.

The case $N = 1$ of a cubic surface is interesting. The line v is not on the surface, and the curve p has to pass through some, or all, of the intersections of v with the surface. The curves u are the plane sections through v. We are in the position, then, of having a 'base' consisting of a complete plane section, a line not on the surface, and a curve which may not be 'basic' in the sense that it could not be included in any base in terms of which all curves on the surface could be expressed. Nevertheless the algebraic expression of the transformations leads to proper curves on the surface, namely the transforms of p, and u.

I proceed now to determine $\mathbf{\Gamma}_v$. Since $u+v$ is a general plane section, $\{u+v, u+v\} = N+2$. We have $\{u,u\} = 0$, $\{u,v\} = 3$, so that $\{v,v\} = N-4$. Also $\{u,p\} = 1$, $\{v,p\} = m-1$, so that, from sym-$\mathbf{P\Gamma}$, $\{p,p\} = -N$. Thus

[6.2] *With base* u, v, p,

$$\mathbf{\Gamma}_v = \begin{pmatrix} 0 & 3 & 1 \\ 3 & N-4 & m-1 \\ 1 & m-1 & -N \end{pmatrix}.$$

Consider next the $\mathscr{R}$-curve $t = \mathscr{P}p$. We have $\mathscr{T}u = u$ and $\mathscr{T}p = p$ (from Theorem [2.1]), so that the first and last rows in the corresponding matrix $\mathbf{T}_v$ are $(1,0,0)$ and $(0,0,1)$. From inv-$\mathbf{T}_v$ and sym-$\mathbf{T}_v\mathbf{\Gamma}$ we find:

[6.3] *With base* u, v, p,

$$[\mathbf{P}p]_v = \mathbf{T}_v = \begin{pmatrix} 1 & 0 & 0 \\ 2(3N+m-1) & -1 & 6 \\ 0 & 0 & 1 \end{pmatrix}.$$

For the rest of this section we use u, t, p as base. The matrices corresponding to this base may be derived from those above, by using the relation $t = (N+2m)u+v-2p$ (from Theorem [6.1]). We find:

[6.4] *With base* u, t $(= \mathscr{P}p)$, p,

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & -N & k \\ 1 & k & -N \end{pmatrix}, \quad \mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 2M & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $k = \{t,p\} = 3N+3m-1$, $M = N+k = 4N+3m-1$.

I wish next to calculate the matrix corresponding to the transformation $[\mathscr{T}t] = [[\mathscr{P}p]\mathscr{P}p]$. We may do this either as in M.I. § 9 (in the calculation of $\mathbf{R}_2$), or by using Theorem [2.2] (i), namely, in the present notation, $[\mathbf{T}t] = \mathbf{PTP}$. The result is:

[6.5] *With base* u, t, p,

$$[\mathbf{T}t] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2M & 2 & -1 \end{pmatrix}.$$

Let ρ, α, β be such that $h = \rho u+\alpha t+\beta p$ is an irreducible $\mathscr{R}$-curve. First, since $\{h,u\} = \rho\{u,u\}+\alpha\{t,u\}+\beta\{p,u\} = 1$, we must have

$$\alpha+\beta = 1. \tag{6.4}$$

The matrix $\mathbf{H}$ associated with the transformation $\mathscr{H}$ will be of the form

$$\mathbf{H} = \rho\mathbf{U}+\alpha\mathbf{E}+\beta\mathbf{F},$$

where $\mathbf{U}$, $\mathbf{E}$, $\mathbf{F}$ are matrices independent of ρ, α, β. Since, when $(\rho, \alpha, \beta) = (0, 1, 0), (0, 0, 1)$, we have $\mathbf{H} = \mathbf{T}, \mathbf{P}$, respectively, we must have $\mathbf{E} = \mathbf{T}$, $\mathbf{F} = \mathbf{P}$. To determine $\mathbf{U}$ we need only consider one other $\mathscr{R}$-curve and transformation; take then $\mathscr{T}t$. From Theorem [6.4], $\mathscr{T}t = 2Mu-t+2p$, and therefore

$$[\mathbf{T}t] = 2M\mathbf{U}-\mathbf{T}+2\mathbf{P},$$

whence, from Theorem [6.5],

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.5}$$

Thus:

[6.6] *If ρ, α, β are such that $h = \rho u+\alpha t+\beta p$ is an $\mathscr{R}$-curve, then the matrix associated with the corresponding $\mathscr{R}$-transformation is*

$$\mathbf{H} = \rho\mathbf{U}+\alpha\mathbf{T}+\beta\mathbf{P},$$

where $\mathbf{U}$ *is given by equation* (6.5), *and* $\mathbf{T}$, $\mathbf{P}$ *by Theorem* [6.4].

We have next to find the conditions satisfied by ρ, α, β in order that h may be an $\mathscr{R}$-curve. We first find conditions that $\mathbf{H}$ may correspond to an involutory self-transformation of π. Since $\mathbf{U\Gamma}$, $\mathbf{T\Gamma}$, $\mathbf{P\Gamma}$ are all symmetrical, a sufficient condition for $\mathbf{H\Gamma H}' = \mathbf{\Gamma}$, is inv-$\mathbf{H}$. Now

$$\mathbf{U}^2 = \mathbf{0}, \qquad \mathbf{UT} = \mathbf{TU} = \mathbf{UP} = \mathbf{PU} = \mathbf{U},$$

and, from $[\mathbf{T}t]^2 = \mathbf{I}$, $\mathbf{PT}+\mathbf{TP} = 2M\mathbf{U}+2\mathbf{I}$, so that

$$\mathbf{H}^2 = (\alpha+\beta)^2\mathbf{I}+\{\rho(\alpha+\beta)+M\alpha\beta\}\mathbf{U}.$$

The conditions are therefore $\alpha+\beta = \pm 1$, where we must take the positive sign since the other gives $\mathscr{H}u = -u$, a meaningless condition geometrically, and

$$\rho = -M\alpha\beta. \tag{6.6}$$

Compare this result with Theorem [2.4], which implies that the matrix corresponding to any $\mathscr{R}$-transformation of the system $\mathscr{R}^1(p)$ is of the form $\mathbf{P}(\mathbf{TP})^n$. We have

$$(\mathbf{PT}-\mathbf{I})^2 = 2M\mathbf{U}, \qquad (\mathbf{PT}-\mathbf{I})^3 = \mathbf{0},$$

and therefore

$$(\mathbf{PT})^n = n(n-1)M\mathbf{U}+n\mathbf{PT}-(n-1)\mathbf{I},$$

so that

$$\mathbf{P}(\mathbf{PT})^\alpha = \rho\mathbf{U}+\alpha\mathbf{T}+\beta\mathbf{P}. \tag{6.7}$$

From Theorem [2.4], the elliptic parameter of the point of a curve u which determines the transformation $\mathscr{P}(\mathscr{T}\mathscr{P})^{-\alpha}$ is $(-3\alpha+1)\phi_p$. Thus

[6.7] (i) *The curve* $h_\alpha = \rho u+\alpha t+\beta p$ *is an* $\mathscr{R}$*-curve if*

$$\alpha+\beta = 1, \qquad \rho = -M\alpha\beta.$$

(ii) *If the elliptic parameters on the curves* u *are determined continuously in such a way that the* $\mathscr{R}$*-curve* p *is the locus of points of parameter* ϕ_p*, then* h_α *is the locus of points of parameter* $(-3\alpha+1)\phi_p$*, i.e.* $\alpha\phi_t+\beta\phi_p$.

(iii) *The matrix corresponding to the* $\mathscr{R}$*-transformation determined by* h_α *is* $\mathbf{H}_\alpha = \mathbf{P}(\mathbf{PT})^\alpha = \rho\mathbf{U}+\alpha\mathbf{T}+\beta\mathbf{P}$.

We have

$$\begin{aligned}\{h_\alpha, v\} &= \rho\{u, v\}+\alpha\{(N+2m)u+v-2p, v\}+\beta\{p, v\}\\ &= \alpha(3\alpha-2)M+m-1,\end{aligned}$$

so that

[6.8] *The order of the* $\mathscr{R}$*-curve* h_α *is* $\alpha(3\alpha-2)M+m$.

In particular, $t = h_1$ is of order $M+m = 4N+4m-1$.

The matrix equation (6.7) is valid (though not the whole truth) for all values of α, not only integers, and the points (or sets of points) $\kappa\phi_p$ on the curves u are definable for all values of κ (as, e.g., in Theorems [2.5] and [2.6]), but a corresponding extension of meaning for the symbolic expression of the curve h_α does not seem possible. Consider the following examples:

(i) $\kappa = 0$, $\alpha = \frac{1}{3}$. This value determines the nine inflexions on each of the curves u. The locus of these inflexions may be proved to be

$$j = (3N+2)u+3v,$$

but

$$h_{\frac{1}{3}} = \tfrac{1}{9}(-5N+2)u+\tfrac{1}{3}v.$$

(ii) $\kappa = \frac{1}{2}$, $\alpha = \frac{1}{2}$. This value determines on each of the curves u the four points of contact of the tangents from the point p_u. The locus of these is the united curve of the involution $\mathscr{P}$, which may be proved to be

$$2d = 2(N+m+1)u+4v-4p,$$

but

$$h_{\frac{1}{2}} = \tfrac{1}{4}(-2N+m+1)u+\tfrac{1}{2}v-\tfrac{1}{2}p.$$

Corresponding extensions of meaning of the matrix equations and the elliptic parameters are possible in the systems $\mathscr{R}^2$ and $\mathscr{R}^r$ discussed in §§ 7, 8, but reference to them is postponed until some more satisfactory account of the geometrical significance of the symbols h_α for non-integer values of α can be given.

7. Matrices associated with a two-fold syzygetic system

Let a, b, c be three $\mathscr{R}$-curves in syzygy, as in § 5, and let their orders be m_0, m_1, m_2, and let $\{b,c\} = k_{12}$, $\{c,a\} = k_{20}$, $\{a,b\} = k_{01}$. Consider on the surface π curves which can be expressed in terms of: u a plane cubic curve, v the multiple line, and b and c. Using the results of Theorem [6.2], we find for the intersection matrix:

[7.1] *With base u, v, b, c*

$$\mathbf{\Gamma}_v = \begin{pmatrix} 0 & 3 & 1 & 1 \\ 3 & N-4 & m_1-1 & m_2-1 \\ 1 & m_1-1 & -N & k_{12} \\ 1 & m_2-1 & k_{12} & -N \end{pmatrix}.$$

Since the curves $\mathscr{B}u$, $\mathscr{B}v$, $\mathscr{B}b$ may be expressed as in Theorem [6.1] independently of the curve c, the first three rows of $\mathbf{B}_v$ are identical with those of $\mathbf{P}_v$, with a column of zeros added; the fourth row may be computed from inv-$\mathbf{B}$ and sym-$\mathbf{B\Gamma}$. We find:

[7.2] *With base u, v, b, c*

$$\mathbf{B}_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ N+2m_1 & 2 & -3 & 0 \\ N+2m_1 & 1 & -2 & 0 \\ m_1+m_2-k_{12} & 1 & -1 & -1 \end{pmatrix}.$$

The matrix $\mathbf{C}_v$ will be symmetrical with this, while to calculate $\mathbf{A}_v$, we may use $\mathscr{A}u = u$, $\mathscr{A}b = c$, $\mathscr{A}c = b$, so that the first, third, and fourth rows are $(1,0,0,0)$, $(0,0,0,1)$, $(0,0,1,0)$. Thus

[7.3] *With base u, v, b, c*

$$\mathbf{A}_v = \begin{pmatrix} 1 & 0 & 0 & 0 \\ (3N+m_1+m_2-3k_{12}-2) & -1 & 3 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

From Theorem [7.2] we see that

$$a = (m_1+m_2-k_{12})u+v-b-c, \tag{7.1}$$

so that $\{a,b\} = (m_1+m_2-k_{12})\{u,b\}+\{v,b\}-\{b,b\}-\{c,b\}$,

i.e. $k_{01} = m_1+m_2-k_{12}+m_2-1+N-k_{12}$,

and there will be two similar relations connecting the orders with the numbers of intersections. These lead to the equations

$$3m_i = 2k_{ip}+2k_{iq}-k_{pq}-N+1, \quad i,p,q = 0,1,2. \tag{7.2}$$

These relations show that $2\sum k_{ij}-N+1$ is a multiple of three.

From equation (7.1) it follows that any curve which may be expressed with integer coefficients in terms of u, v, b, c may also be so expressed in terms of u, a, b, c. We may find the matrices associated with this base either by transforming the equations above, using equation (7.1), or directly from the properties contained in Theorems [5.2] and [1.1]. The result is:

[7.4] *With base* u, a, b, c,

$$\mathbf{\Gamma} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & -N & k_{01} & k_{02} \\ 1 & k_{10} & -N & k_{12} \\ 1 & k_{20} & k_{21} & -N \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ N_0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ N_1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ N_2 & 1 & 1 & -1 \end{pmatrix},$$

where $N_i = N + k_{ip} + k_{iq} - k_{pq}$.

From the matrix $\mathbf{A}$ we have

$$\mathscr{A}a = N_0 u - a + b + c,$$

and therefore there is a matrix $\mathbf{U}$ for which

$$[\mathbf{A}a] = N_0 \mathbf{U} - \mathbf{A} + \mathbf{B} + \mathbf{C}.$$

We have $[\mathscr{A}a]u = u$, $[\mathscr{A}a]a = a$, giving the first two rows of $[\mathbf{A}a]$. The others may be found from inv-$[\mathbf{A}a]$, and sym-$[\mathbf{A}a]\mathbf{\Gamma}$. The result is

$$[\mathbf{A}a] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2N + 2k_{01} & 2 & -1 & 0 \\ 2N + 2k_{02} & 2 & 0 & -1 \end{pmatrix},$$

whence

$$\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

If $h = \rho u + \alpha a + \beta b + \gamma c$ is an $\mathscr{R}$-curve, then the matrix of the corresponding $\mathscr{R}$-transformation is

$$\mathbf{H} = \rho \mathbf{U} + \alpha \mathbf{A} + \beta \mathbf{B} + \gamma \mathbf{C}.$$

First, since $\{h, u\} = 1$, we have

$$\alpha + \beta + \gamma = 1. \tag{7.4}$$

Next, since sym-$\mathbf{U\Gamma}$, and sym-$\mathbf{A\Gamma}$, etc., we have sym-$\mathbf{H\Gamma}$, and consequently we have only to investigate the condition inv-$\mathbf{H}$. This yields, in addition to equation (7.4), the single condition

$$-\rho = \beta\gamma(N+k_{12})+\gamma\alpha(N+k_{20})+\alpha\beta(N+k_{01}),$$

which may also be written

$$-\rho = \tfrac{1}{2}(N+\boldsymbol{\alpha}'\mathbf{K}\boldsymbol{\alpha}), \tag{7.5}$$

where $\boldsymbol{\alpha}'$ is the row vector (α, β, γ) and $\mathbf{K}$ is the matrix formed by the last three rows and columns of $\mathbf{\Gamma}$. Thus

[7.5] *$h = \rho u+\alpha a+\beta b+\gamma c$ is an $\mathscr{R}$-curve if equations* (7.4) *and* (7.5) *are satisfied; the matrix of the corresponding $\mathscr{R}$-transformation is*

$$\mathbf{H} = \rho\mathbf{U}+\alpha\mathbf{A}+\beta\mathbf{B}+\gamma\mathbf{C},$$

$\mathbf{U}$ *being given by equation* (7.3).

Consider next $(\mathbf{AB})^\beta$. By direct multiplication we find

$$(\mathbf{AB}-\mathbf{I})^2 = (N_0+N_1)\mathbf{U}, \qquad (\mathbf{AB}-\mathbf{I})^3 = \mathbf{0},$$

so that $$(\mathbf{AB})^\beta = (1-\beta)\mathbf{I}+\beta\mathbf{AB}+\tfrac{1}{2}\beta(\beta-1)(N_0+N_1)\mathbf{U},$$

and similarly

$$(\mathbf{AC})^\gamma = (1-\gamma)\mathbf{I}+\gamma\mathbf{AC}+\tfrac{1}{2}\gamma(\gamma-1)(N_0+N_2)\mathbf{U}.$$

Thence, using

$$\mathbf{BAC} = [\mathbf{A}a] = N_0\mathbf{U}-\mathbf{A}+\mathbf{B}+\mathbf{C} \text{ (Theorem [2.2] (ii))},$$

we find

$$\begin{aligned}\mathbf{A}(\mathbf{AB})^\beta(\mathbf{AC})^\gamma &= \tfrac{1}{2}\{\alpha(\alpha-1)N_0+\beta(\beta-1)N_1+\gamma(\gamma-1)N_2\}\mathbf{U}+\alpha\mathbf{A}+\beta\mathbf{B}+\gamma\mathbf{C}\\ &= \rho\mathbf{U}+\alpha\mathbf{A}+\beta\mathbf{B}+\gamma\mathbf{C},\end{aligned}$$

so that

[7.6] *The $\mathscr{R}$-transformation $\mathscr{A}(\mathscr{AB})^\beta(\mathscr{AC})^\gamma$ is that determined by the $\mathscr{R}$-curve $\rho u+\alpha a+\beta b+\gamma c$, where ρ, α, β, γ are connected by equations* (7.4) *and* (7.5).

Using Theorem [3.4], we find from this:

[7.7] *If a, b, c cut a curve u in points of elliptic parameters ϕ_a, ϕ_b, ϕ_c, then the $\mathscr{R}$-curve $\rho u+\alpha a+\beta b+\gamma c$ cuts it in the point $\alpha\phi_a+\beta\phi_b+\gamma\phi_c$.*

8. Matrices associated with an r-fold syzygetic system

We found in § 4 that, if $r > 2$, it is not possible to select a symmetrical set of $r+1$ transformations from the r-fold system $\mathscr{S}^r(a_i)$

so that the simple form of matrices set out in Theorem [7.4] cannot be adapted to higher values of r. Initially it appears best to use the obvious base consisting of the curves u, v, $a_1,\ldots,a_r$, the matrices corresponding to which may be written down by extension from those in Theorems [7.1] and [7.2]. To express them in concise form I use the following notation for row vectors with r elements corresponding to the subscripts $1,\ldots,r$:

$$\mathbf{o}' = (0,0,\ldots,0), \qquad \boldsymbol{\upsilon}' = (1,1,\ldots,1),$$

$\boldsymbol{\epsilon}_i' = (0,0,\ldots,0,1,0,\ldots,0)$ (all elements zero except that corresponding to the subscript i),

$$\mathbf{m}' = (m_1,\ldots,m_r), \qquad \mathbf{k}_i' = (k_{1i},k_{2i},\ldots,k_{ri}), \quad \text{with} \quad k_{ii} = -N,$$

and write $\mathbf{o},\boldsymbol{\upsilon}$, etc., for the corresponding column vectors. Also put

$$\mathbf{K} = (k_{ij}), \quad \text{with} \quad k_{ii} = -N \quad (i,j = 1,\ldots,r).$$

We find:

[8.1] *With base* u, v, a_i

$$\boldsymbol{\Gamma}_v = \begin{pmatrix} 0 & 3 & \boldsymbol{\upsilon}' \\ 3 & N-4 & \mathbf{m}'-\boldsymbol{\upsilon}' \\ \boldsymbol{\upsilon} & \mathbf{m}-\boldsymbol{\upsilon} & \mathbf{K} \end{pmatrix},$$

$$\mathbf{A}_{i,v} = \begin{pmatrix} 1 & 0 & \mathbf{o}' \\ N+2m_i & 2 & -3\boldsymbol{\epsilon}_i' \\ m_i\boldsymbol{\upsilon}+\mathbf{m}-\mathbf{k}_i & \boldsymbol{\upsilon} & -\boldsymbol{\upsilon}\boldsymbol{\epsilon}_i'+\mathbf{I} \end{pmatrix}.$$

As in previous cases, if $h = \rho u+\sigma v+\sum \alpha_i a_i$ is an $\mathscr{R}$-curve, then, for some constant matrices $\mathbf{U}$, $\mathbf{V}$, the matrix of the corresponding $\mathscr{R}$-transformation is $\mathbf{H} = \rho\mathbf{U}+\sigma\mathbf{V}+\sum \alpha_i \mathbf{A}_i$, where, since $\{h,u\} = 1$ (or, since $\mathscr{H}u = u$, making the first row of $\mathbf{H}$ $(1,0,0,\ldots,0)$), we have

$$3\sigma+\sum \alpha_i = 1. \tag{8.1}$$

To find $\mathbf{U}$ and $\mathbf{V}$ we have to calculate the matrices for two further transformations, say $\mathscr{A}_{11}$ and $\mathscr{A}_{12}$. From Theorem [8.1], we have

$$\left.\begin{aligned} \mathbf{A}_{ii} &= [\mathbf{A}_i a_i] = (N+2m_i)\mathbf{U}+\mathbf{V}-2\mathbf{A}_i \\ \mathbf{A}_{ij} &= [\mathbf{A}_i a_j] = (m_i+m_j-k_{ij})\mathbf{U}+\mathbf{V}-\mathbf{A}_i-\mathbf{A}_j \end{aligned}\right\}. \tag{8.2}$$

The elements in the first four rows and columns of $\mathbf{A}_{11}$ and $\mathbf{A}_{12}$ were found in § 7, and the rest may be determined from inv-$\mathbf{A}$ and sym-$\mathbf{A}\boldsymbol{\Gamma}$.

The result is:

[8.2] *With base u, v, a_i, the matrix of the $\mathscr{R}$-transformation corresponding to the $\mathscr{R}$-curve $\rho u+\sigma v+\sum \alpha_i a_i$ is $\rho\mathbf{U}+\sigma\mathbf{V}+\sum \alpha_i \mathbf{A}_i$, where*

$$\mathbf{U}=\begin{pmatrix}0 & 0 & \mathbf{o}'\\ 3 & 0 & \mathbf{o}'\\ \mathbf{o} & \mathbf{o} & \mathbf{0}\end{pmatrix},\qquad \mathbf{V}=\begin{pmatrix}3 & 0 & \mathbf{o}'\\ 5N-2 & 3 & \mathbf{o}'\\ N\mathbf{o}+2\mathbf{m} & 2\mathbf{o} & -3\mathbf{I}\end{pmatrix}.$$

The products of the pairs of matrices are

$$\left.\begin{aligned}&\mathbf{U}^2=\mathbf{0},\quad \mathbf{UV}=\mathbf{VU}=3\mathbf{U},\quad \mathbf{UA}_i=\mathbf{A}_i\mathbf{U}=\mathbf{U}\\ &\mathbf{V}^2=2(5N-2)\mathbf{U}+9\mathbf{I},\quad \mathbf{A}_i^2=\mathbf{I}\end{aligned}\right\},\tag{8.3}$$

and, from $\mathbf{A}_{ii}^2=\mathbf{I}$, $\mathbf{A}_{ij}^2=\mathbf{I}$,

$$\left.\begin{aligned}&\mathbf{VA}_i+\mathbf{A}_i\mathbf{V}=(6N+2m_i-2)\mathbf{U}+6\mathbf{I}\\ &\mathbf{A}_i\mathbf{A}_j+\mathbf{A}_j\mathbf{A}_i=(2N+2k_{ij})\mathbf{U}+2\mathbf{I}\end{aligned}\right\}.\tag{8.4}$$

Again $\mathbf{U\Gamma}$, $\mathbf{V\Gamma}$, and $\mathbf{A}_i\mathbf{\Gamma}$ are symmetrical, so that $\mathbf{H\Gamma}$ is symmetrical. From $\mathbf{H}^2=\mathbf{I}$ we find, in addition to equation (8.1),

$$\left.\begin{aligned}\rho&=-\tfrac{1}{2}\{\sigma^2(N-4)+2\sigma\textstyle\sum\alpha_i(m_i-1)+\boldsymbol{\alpha}'\mathbf{K}\boldsymbol{\alpha}+N\}\\ &=-\tfrac{1}{2}\{\boldsymbol{\alpha}^{*\prime}\mathbf{K}^*\boldsymbol{\alpha}^*+N\},\end{aligned}\right\},\tag{8.5}$$

where $\boldsymbol{\alpha}^{*\prime}=(\sigma,\alpha_1,\ldots,\alpha_r)$ and $\mathbf{K}^*$ is the matrix obtained when the first row and column of $\mathbf{\Gamma}$ are removed. We have proved that:

[8.3] *The curve $\rho u+\sigma v+\sum \alpha_i a_i$ is an $\mathscr{R}$-curve for all values of ρ, σ, α_i which satisfy equations* (8.1) *and* (8.5).

The following relations may be proved:

$$\left.\begin{aligned}&\mathbf{A}_p\mathbf{A}_i\mathbf{A}_q=\mathbf{A}_q\mathbf{A}_i\mathbf{A}_p=(N+k_{ip}+k_{iq}-k_{pq})\mathbf{U}-\mathbf{A}_i+\mathbf{A}_p+\mathbf{A}_q\\ &\mathbf{A}_p\mathbf{VA}_q=\mathbf{A}_q\mathbf{VA}_p=(3N+m_p+m_q-3k_{pq}-2)\mathbf{U}-\mathbf{V}+3\mathbf{A}_p+3\mathbf{A}_q\\ &\mathbf{A}_p\mathbf{A}_q\mathbf{V}=\mathbf{VA}_q\mathbf{A}_p=(3N-m_p+m_q+3k_{pq})\mathbf{U}+\mathbf{V}+3\mathbf{A}_p-3\mathbf{A}_q\end{aligned}\right\}.\tag{8.6}$$

These equations are valid when $p=q$, if k_{pp} and k_{qq} are set equal to $-N$. The first members of equations (8.6) show that the product of any odd number of factors $\mathbf{A}_1,\ldots,\mathbf{A}_r,\mathbf{V}$ is unaltered when the order of any odd number of consecutive factors is reversed. Further $\mathbf{V\Gamma}$ is symmetrical. On the other hand, $\mathbf{V}$ is not involutory (so that $\mathbf{V\Gamma V}'\neq\mathbf{\Gamma}$), and therefore $\mathbf{V}$ cannot be used as the matrix to complete the symmetrical set of $r+1$ matrices.

Consider now

$$\begin{aligned}\mathbf{A}_p\mathbf{HA}_q&=\rho\mathbf{A}_p\mathbf{UA}_q+\sigma\mathbf{A}_p\mathbf{VA}_q+\textstyle\sum\alpha_i\mathbf{A}_p\mathbf{A}_i\mathbf{A}_q\\ &=\rho\mathbf{U}+\sigma\{(3N+m_p+m_q-3k_{pq}-2)\mathbf{U}-\mathbf{V}+3\mathbf{A}_p+3\mathbf{A}_q\}+\\ &\qquad+\textstyle\sum\alpha_i\{(N+k_{ip}+k_{iq}-k_{pq})\mathbf{U}-\mathbf{A}_i+\mathbf{A}_p+\mathbf{A}_q\}\\ &=\bar\rho\mathbf{U}+\bar\sigma\mathbf{V}+\textstyle\sum\bar\alpha_i\mathbf{A}_i,\end{aligned}$$

where, using equation (8.1), we have

$$\bar{\sigma} = -\sigma,$$

$$\bar{\alpha}_i = -\alpha_i \quad (i \neq p, q),$$

$$\bar{\alpha}_p = -\alpha_p + 1, \qquad \bar{\alpha}_q = -\alpha_q + 1,$$

$$\bar{\rho} = -\tfrac{1}{2}\{\bar{\boldsymbol{\alpha}}^{*\prime}\mathbf{K}^*\bar{\boldsymbol{\alpha}}^* + N\},$$

i.e. $$\mathbf{A}_p\mathbf{H}\mathbf{A}_q = -\mathbf{H} + \mathbf{A}_p + \mathbf{A}_q + (\bar{\rho}+\rho)\mathbf{U}.$$

The matrix $\mathbf{H}$ is uniquely determined by the numbers $\alpha_1, \ldots, \alpha_r$, so that we may compare this result with Theorem [1.5]. Thence it follows by induction that:

[8.4] *If the curves a_i cut any of the curves u in points of elliptic parameters ϕ_i, then the $\mathscr{R}$-curve $\rho u + \sigma v + \sum \alpha_i a_i$ cuts u in the point of parameter $\sum \alpha_i \phi_i$.*

We may now write

$$\mathbf{H} = \mathbf{H}_{\alpha_1 \ldots \alpha_r} = \mathbf{H}_{\boldsymbol{\alpha}} = \rho\mathbf{U} + \sigma\mathbf{V} + \sum \alpha_i \mathbf{A}_i,$$

with, in particular,

$$\mathbf{A}_i = \mathbf{H}_{\boldsymbol{\epsilon}_i}, \qquad \mathbf{A}_{ij} = \mathbf{H}_{-\boldsymbol{\epsilon}_i - \boldsymbol{\epsilon}_j}.$$

Then, corresponding to Theorem [4.6] we have:

[8.5] *The matrices $\mathbf{H}_{\boldsymbol{\alpha}}$ are the products of odd numbers of the matrices $\mathbf{H}_{\boldsymbol{\epsilon}_1}, \ldots, \mathbf{H}_{\boldsymbol{\epsilon}_r}$ and $\mathbf{H}_{-2\boldsymbol{\epsilon}_1}$. They form a system in which the product of any three is given by $\mathbf{H}_{\boldsymbol{\alpha}}\mathbf{H}_{\boldsymbol{\beta}}\mathbf{H}_{\boldsymbol{\gamma}} = \mathbf{H}_{\boldsymbol{\alpha}-\boldsymbol{\beta}+\boldsymbol{\gamma}}$.*

9. The algebraic content of the results

In this section I use $\boldsymbol{\upsilon}$, $\boldsymbol{\epsilon}_i$, $\mathbf{K}$, etc., with the same significance as in § 8, but regard the vectors as having $r+1$ components, with subscripts $0, 1, \ldots, r$. The results proved in §§ 6–8 may be taken to give a solution to the problem:

$\boldsymbol{\Gamma}$ is a given symmetrical matrix of $r+2$ rows and columns of the form

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & \boldsymbol{\upsilon}' \\ \boldsymbol{\upsilon} & \mathbf{K} \end{pmatrix}, \tag{9.1}$$

where $\mathbf{K}$ is a symmetrical matrix whose elements are unrestricted except that

$$k_{00} = k_{11} = \ldots = k_{rr} = -N. \tag{9.2}$$

We are required to find a set of $r+1$ independent matrices $\mathbf{X}_0, \mathbf{X}_1, \ldots, \mathbf{X}_r$

which determine an r-fold syzygetic system, namely a system of matrices $\mathbf{Y}$ *such that*

(i) $\mathbf{Y}^2 = \mathbf{I}$,

(ii) $\mathbf{Y\Gamma Y}' = \mathbf{\Gamma}$, *i.e. with* (i), $\mathbf{Y\Gamma}$ *is symmetrical,*

(iii) $\mathbf{Y}$ *may be expressed in the form*

$$\mathbf{Y}_\alpha = \rho\mathbf{U} + \sum \alpha_i \mathbf{X}_i,$$

where

$$\mathbf{U} = \begin{pmatrix} 0 & \mathbf{o}' \\ \mathbf{o} & \mathbf{0} \end{pmatrix},$$

(iv) *the product of any three of the matrices is another matrix of the system, namely* $\mathbf{Y}_\alpha \mathbf{Y}_\beta \mathbf{Y}_\gamma = \mathbf{Y}_{\alpha-\beta+\gamma}$.

So long as we confine our attention to the geometric interpretation of the results, there are restrictions on the k_{ij} in addition to equations (9.2). One such is contained in equation (7.2). Another is that the signature of $\mathbf{\Gamma}$ must be $(1, r+1)$ [Hodge, 1937]. Again the value of N must be sufficiently high to allow a given number r of $\mathscr{R}$-curves to be forced on to the surface. On the other hand, the relations obtained in §§ 6–8 are purely algebraic, and involve no restrictions on the k_{ij} other than equations (9.2); the k_{ij} are not even confined to being rational or real, but, if they are integers, then the elements of all the matrices $\mathbf{Y}$ are integers. The solutions for the cases $r = 1, 2$ are simple, and are contained in Theorems [6.4], [6.7], [7.4], [7.5], [7.6]. For a general value of r, as we have seen, it is not possible to select a symmetrical set of $r+1$ curves, or transformations, or matrices. We may, however, reduce the matrix $\mathbf{\Gamma}_v$ of Theorem [8.1] to the symmetrical form given in equation (9.1) by replacing the base u, v, a_i by u, x_0, x_i, where

$$x_0 = \mathscr{A}_1 a_1 = (N+2m_1)u + v - 2a_1 \quad \text{(cf. equation (8.2))}$$

$$x_i = a_i \quad (i \neq 0).$$

In this system the subscripts 0, 1 behave differently from the remaining subscripts, but, as we see below, all the matrices may be expressed in the same symbolic form.

Since x_0 is an $\mathscr{R}$-curve,

$$k_{00} = -N,$$

and, as in equation (7.2),

$$k_{0i} = N + 2m_1 + m_i - 2k_{1i} - 1.$$

Effecting this transformation, we find $\mathbf{\Gamma}$ in the required form (9.1), and

$$\mathbf{X}_i = \begin{pmatrix} 1 & \mathbf{o}' \\ \mathbf{d}_i & \{\upsilon(\boldsymbol{\epsilon}_0'+2\boldsymbol{\epsilon}_1'-\boldsymbol{\epsilon}_i')-\mathbf{I}\} \end{pmatrix} \quad (i = 0, 1, \ldots, r),$$

where the columns $\mathbf{d}_i$ form a symmetrical matrix $\mathbf{D} = (d_{ij})$, in which

$$\begin{aligned} d_{ji} = d_{ij} &= N_{01,i}+N_{01,j}+N_{ij,1}, \\ N_{qp,i} &= N_{pq,i} = N+k_{ip}+k_{iq}-k_{pq}, \\ N_{ip,i} &= N_{pi,i} = 2N+2k_{ip}, \\ N_{ii,p} &= 0. \end{aligned}$$

These are the required $r+1$ matrices $\mathbf{X}_i$; in terms of them we have

$$\begin{aligned} \mathbf{Y}_{\alpha_0\,\alpha_1\ldots\alpha_r} &= \mathbf{X}_0(\mathbf{X}_0\mathbf{X}_1)^{\alpha_1}(\mathbf{X}_0\mathbf{X}_2)^{\alpha_2}\ldots(\mathbf{X}_0\mathbf{X}_r)^{\alpha_r}, \\ &= \rho\mathbf{U}+\sum_0^r \alpha_i\mathbf{X}_i \\ &= \begin{pmatrix} 1 & \mathbf{o}' \\ \rho\upsilon+\mathbf{D}\boldsymbol{\alpha} & \{\upsilon(\boldsymbol{\epsilon}_0'+2\boldsymbol{\epsilon}_1'-\boldsymbol{\alpha}')-\mathbf{I}\} \end{pmatrix}, \end{aligned}$$

where $$\sum_0^r \alpha_i = 1,$$

and $$\rho = -\tfrac{1}{2}(\boldsymbol{\alpha}'\mathbf{K}\boldsymbol{\alpha}+N),$$

i.e. $$\boldsymbol{\alpha}^{*\prime}\mathbf{\Gamma}\boldsymbol{\alpha}^*+N = 0,$$

where $$\boldsymbol{\alpha}^{*\prime} = (\rho, \alpha_0, \alpha_1, \ldots, \alpha_r).$$

REFERENCES

F. Enriques, *Rend. R. Accad. dei Lincei* (5), 15 (1906), 665.
—— ibid. (5), 21 (1912), 14.
W. V. D. Hodge, *J. of London Math. Soc.* 12 (1937), 58.
A. R. Richardson, *Proc. London Math. Soc.* (2), 47 (1942), 38.
T. G. Room, *Proc. Royal Soc.* A. 193 (1948), 25 (quoted as M.I.).
F. Severi, *Math. Ann.* 62 (1906), 194.
—— *Rend. Circ. Mat. Palermo* 30 (1911), 265.
V. Snyder and F. R. Sharpe, *Trans. American Math. Soc.* 15 (1914), 266.
—— ibid. 16 (1915), 62.

[*Note added 20 October 1948*]

A simple geometric representation of the continuous groupoid†

An involutory unitary $(n+1)\times(n+1)$ matrix $\mathbf{H}$ may be regarded as determining an involutory collineation $\mathscr{H}$ of a projective space

† This note is the outcome of some suggestions made by Professor H. F. Baker.

of n dimensions in which the base elements are an $[r]$ and an $[n-r-1]$ which are polars with regard to the fundamental quadric primal. If, using $x_0,\ldots,x_n$ as coordinates, we take the quadric to be

$$\mathbf{x}'\boldsymbol{\Gamma}\mathbf{x} = 0,$$

so that

$$\mathbf{H}\boldsymbol{\Gamma}\mathbf{H}' = \boldsymbol{\Gamma},$$

then $\mathscr{H}$ is the transformation in which the transform of the prime $\mathbf{p}'\mathbf{x} = 0$ is the prime

$$\mathbf{p}'\mathbf{H}\mathbf{x} = 0.$$

The base elements of the inversion are the self-corresponding elements of the system, and are therefore given by the latent roots of the matrix $\mathbf{H}$.

For the matrix $\mathbf{Y}$ we find

$$|\mathbf{Y}-\theta\mathbf{I}| = (\theta-1)^2(\theta+1)^r,$$

so that $\mathbf{Y}$ corresponds to a harmonic inversion with regard to a line and an $[r-1]$ in $[r+1]$.

To obtain the complete geometrical interpretation in this sense of the system of matrices $\mathbf{Y}$, I make the following simplifications in the notation:

(i) change the names of the coordinates, so that the first rows of the matrices correspond to z_0, while the rows previously carrying subscripts $0, 1,\ldots, r$ correspond to $z_1, z_2,\ldots, z_n$ $(n = r+1)$; denote by $Z_0, Z_1,\ldots, Z_n$ the vertices of the simplex of reference, and by $\zeta_0, \zeta_1,\ldots, \zeta_n$ the tangential coordinates corresponding to them;

(ii) write $\lambda_1 = 1-\alpha_0$, $\lambda_2 = 2-\alpha_1$, $\lambda_3 = -\alpha_2,\ldots$, $\lambda_n = -\alpha_r$, and allow the λ_i to vary continuously in the field of complex numbers.

The matrix $\boldsymbol{\Gamma}$ is now

$$\boldsymbol{\Gamma} = \begin{pmatrix} 0 & \boldsymbol{\upsilon}' \\ \boldsymbol{\upsilon} & \mathbf{K} \end{pmatrix},$$

where $\mathbf{K} = \mathbf{K}' = (k_{ij})$ $(i,j = 1,\ldots,n)$; the fundamental quadric is

$$Q \equiv \mathbf{z}'\boldsymbol{\Gamma}\mathbf{z} \equiv 2z_0\sum_1^n z_i + \sum_1^n k_{ij}z_iz_j = 0,$$

and the matrix $\mathbf{Y}$ is replaced by

$$\mathbf{H}_{\boldsymbol{\lambda}} = \begin{pmatrix} 1 & \mathbf{o}' \\ \mathbf{K}\boldsymbol{\lambda}-\kappa_{\boldsymbol{\lambda}}\boldsymbol{\upsilon} & \boldsymbol{\upsilon}\boldsymbol{\lambda}'-\mathbf{I}_n \end{pmatrix},$$

where

$$\sum_1^n \lambda_i = 2,$$

$$\kappa_{\boldsymbol{\lambda}} = \tfrac{1}{2}\boldsymbol{\lambda}'\mathbf{K}\boldsymbol{\lambda} = \tfrac{1}{2}\sum k_{ij}\lambda_i\lambda_j.$$

The elements of the first column of $\mathbf{H}_{\boldsymbol{\lambda}}$, the relation connecting the λ_i, and the value of $\kappa_{\boldsymbol{\lambda}}$ are determined completely from the remaining elements of the matrix by the conditions inv-$\mathbf{H}_{\boldsymbol{\lambda}}$ and sym-$\mathbf{H}_{\boldsymbol{\lambda}}\boldsymbol{\Gamma}$. The specialization of $\mathbf{K}$ involved in the equality of the elements of the leading diagonal (namely $k_{ii} = -N$, $-N$ being the number of virtual self-intersections of each of the base curves on the surface) is no longer necessary.

The matrices $\mathbf{H}_{\boldsymbol{\lambda}}$ have the simple algebraic property

$$\mathbf{H}_{\boldsymbol{\lambda}}\mathbf{H}_{\boldsymbol{\mu}} = \begin{pmatrix} 1 & \mathbf{o}' \\ \mathbf{K}(\boldsymbol{\lambda}-\boldsymbol{\mu})-\kappa_{\boldsymbol{\lambda}-\boldsymbol{\mu}}\boldsymbol{\upsilon} & \boldsymbol{\upsilon}(\boldsymbol{\lambda}'-\boldsymbol{\mu}')+\mathbf{I}_n \end{pmatrix},$$

i.e. the products of the matrices determined by the sets of parameters λ_i, μ_i depend only on the differences $\lambda_i-\mu_i$. This is equivalent to the groupoid property

$$\mathbf{H}_{\boldsymbol{\lambda}}\mathbf{H}_{\boldsymbol{\mu}}\mathbf{H}_{\boldsymbol{\nu}} = \mathbf{H}_{\boldsymbol{\lambda}-\boldsymbol{\mu}+\boldsymbol{\nu}}$$

set out below.

The $(n-1)$-fold root $\theta = -1$ of the characteristic equation of $\mathbf{H}_{\boldsymbol{\lambda}}$ determines the self-corresponding line in the harmonic inversion: the tangential equations of this line are therefore

$$(\mathbf{H}_{\boldsymbol{\lambda}}+\mathbf{I}_{n+1})\boldsymbol{\zeta} = \mathbf{0}.$$

These all depend linearly on the two equations

$$\zeta_0 = 0, \qquad \sum_1^n \lambda_i \zeta_i = 0.$$

Thus the self-corresponding line of $\mathscr{H}_{\boldsymbol{\lambda}}$ is the line which joins Z_0 to the point $\sum_1^n \lambda_i Z_i$. As $\lambda_i,\ldots,\lambda_n$ are varied, we obtain the ∞^{n-1} family of harmonic inversions determined by the complete family of lines passing through Z_0 and their polar $[n-2]$'s with regard to Q, all of these $[n-2]$'s lying in the prime $\sum_1^n z_i = 0$. Further Z_0 lies on Q, and $\sum z_i = 0$ is the tangent prime at the point. The families of lines and $[n-2]$'s, and therefore also of harmonic inversions, are completely determined by the (general) quadric Q, and a point on it. Thus

[N. 1] *The family of involutory collineations of* $[n]$ *determined by lines through a point and the polar* $[n-2]$*'s of these lines with regard to a quadric which passes through this point forms a groupoid in which the product of any three members of the groupoid is a member of the groupoid.*

Explicitly, if the point is Z_0, the tangent prime to the quadric at that point is $\sum_1^n z_i = 0$, and the lines are specified as the joins of Z_0 to points of $z_0 = 0$, then the product of the collineations specified by the points $\sum \lambda_i Z_i$, $\sum \mu_i Z_i$, $\sum \nu_i Z_i$ in that order is the collineation specified by the point $\sum (\lambda_i - \mu_i + \nu_i) Z_i$. The points corresponding to the transformations of the original system $\mathscr{S}^{n-1}$ are those of which the (homogeneous) coordinates λ_i may be expressed as integers with sum 2.

ON PARALLEL FIELDS OF PLANES IN A RIEMANNIAN SPACE

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1. Introduction

PARALLEL fields of partially null vector spaces in a Riemannian V_n have been discussed in a general way by A. G. Walker.* Some of his results are stated or obtained afresh in §§ 1, 2 of this paper, and the terminology is largely his. The later sections are devoted to a detailed analysis for the case $n = 4$, as a preliminary to a study of the general case, which is reserved for a later paper.

The Riemannian spaces considered in this paper are assumed to be analytic, and not necessarily of positive-definite metric. The fundamental tensor will be denoted by g_{ij}.

In the $(n-1)$-dimensional projective space S_{n-1} associated with any given point P, (x^i), of V_n, the components X^i of a non-zero vector at P are homogeneous coordinates of a point. Those of a covariant vector u_i are homogeneous coordinates of the $(n-2)$-flat† of equation $u_i X^i = 0$. Every contravariant null vector ξ^i at P satisfies the equation $g_{ij}\xi^i\xi^j = 0$, and therefore corresponds in S_{n-1} to a point on the *fundamental hyperquadric* of equation $g_{ij}X^iX^j = 0$. Perpendicular vectors X^i, Y^i at P satisfy $g_{ij}X^iY^j = 0$, and so correspond to points of S_{n-1} that are conjugate with respect to the hyperquadric. Similar remarks apply to covariant vectors, the tangential equation of the hyperquadric being $g^{ij}u_iu_j = 0$.

2. Parallel r-planes

A set of r independent contravariant vector fields‡ $\lambda^i_\alpha \equiv (\lambda^i_1,\ldots,\lambda^i_r)$ in V_n, where $0 \leqslant r \leqslant n$, defines at every point P a (contravariant) vector space $\mathbf{V}^r(P)$ for which, at that point, they form a basis. If P, Q

* See above, pp. 135–45.

† A linear space of p dimensions in S_{n-1} will be called a 'p-flat'. In previous papers I have called such a space a 'p-plane', but have now changed the terminology because 'p-plane' is used in a different sense by Walker. A 'p-plane' is defined below in § 2.

‡ Throughout the paper h, i, j,... will be used for tensor suffixes. Greek letters will be used as ordinal suffixes and will take various ranges as indicated. For the moment they run from 1 to r.

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are any two points of V_n, and if, under parallel transport along any curve joining P, Q, every vector of $\mathbf{V}^r(P)$ goes into a vector of $\mathbf{V}^r(Q)$, then the field of vector spaces having the λ_α^i as basis is said to be *parallel*, and is called a *parallel field of r-planes*. I shall usually refer to it as a *parallel r-plane*. A necessary and sufficient condition that a set of independent vector fields λ_α^i should define a parallel r-plane is that their covariant derivatives $\lambda_{\alpha,j}^i$ should be a linear combination of the λ_α^i, say

$$\lambda_{\alpha,j}^i = A_{\alpha j}^\beta \lambda_\beta^i, \tag{2.1}$$

where the $A_{\alpha j}^\beta$ ($\alpha, \beta = 1, 2, \ldots, r$) are covariant vector fields of V_n [Walker, loc. cit., Theorem 3.3]. The relations (2.1) will be called *recurrence-formulae* for the λ's.

A change of basis $\lambda \to \mu$ for the field of r-planes is defined by linear equations $\mu_\alpha^i = t_\alpha^\beta \lambda_\beta^i$, where the t's are scalars of non-zero determinant. It is obvious that the μ_α^i satisfy equations of the form (2.1). Moreover, if $r > 1$, the t_α^β can be chosen, in an infinite number of ways, so as to make the new base-vectors μ_α^i mutually orthogonal, and, when non-null, of unit length. The μ_α^i are then said to form a *normal* basis.

For a given parallel r-plane, the number s of null vectors is the same for every normal basis, and

$$s \leqslant \min(r, n-r). \tag{2.2}$$

These facts are obvious from the geometry of S_{n-1}. For, supposing $r \neq 0$, the λ_α^i are coordinates of r independent points of S_{n-1}, and therefore define an $(r-1)$-flat S_{r-1}. A change $\lambda \to \mu$ to a normal basis corresponds to the selection of another r independent points μ_α^i in S_{r-1}, these being mutually conjugate with respect to the fundamental hyperquadric because the vectors μ_α^i of V_n are mutually orthogonal.

If S_{r-1} does not touch the hyperquadric, none of the conjugate points μ_α^i lies upon it, and the corresponding vectors of V_n are all non-null. In this case $s = 0$, and (2.2) is obviously true.

If, on the other hand, S_{r-1} touches the hyperquadric in an $(s-1)$-flat, say, then s of the mutually conjugate points μ_α^i lie in this $(s-1)$-flat and hence upon the hyperquadric. Therefore they correspond to s mutually orthogonal null vectors of V_n. The remaining $r-s$ of the points μ_α^i lie elsewhere in S_{r-1}, not upon the hyperquadric, and correspond to $r-s$ non-null vectors of V_n. The invariance of s under a change of normal basis follows at once, because s of the

points of *any* mutually conjugate set in S_{r-1} lie upon the fundamental hyperquadric; while the inequality (2.2) is also an immediate consequence of well-known properties of hyperquadrics.*

The $(s-1)$-flat of contact of S_{r-1} will be called the *null part* of S_{r-1}. It corresponds in V_n to the null part, in Walker's sense, of the vector space $\mathbf{V}^r$. If $s = 0$, S_{r-1} and the corresponding $\mathbf{V}^r$ are *non-null*, this being the case, mentioned above, when S_{r-1} does not touch the hyperquadric. If $s \leqslant r$, S_{r-1} and $\mathbf{V}^r$ are *partially null*. If $s = r$, so that, by (2.2), $r \leqslant \frac{1}{2}n$, S_{r-1} lies wholly in the hyperquadric, and the vectors of any basis are therefore all null and mutually orthogonal. In this case S_{r-1} and the corresponding $\mathbf{V}^r$ are *null*. It will be noted that null planes are included in the class of partially null planes.

In every case, s will be called the *nullity*† of the r-plane $\mathbf{V}^r$ or of the corresponding $(r-1)$-flat S_{r-1}.

In S_{n-1}, the polar of the $(r-1)$-flat S_{r-1} with respect to the fundamental hyperquadric is an $(n-r-1)$-flat, S_{n-r-1} say, which touches the hyperquadric in the same $(s-1)$-flat as S_{r-1}. Thus, incidentally, S_{r-1} and S_{n-r-1} intersect in this $(s-1)$-flat, which is therefore their common null part. In V_n the $(n-r)$-plane $\mathbf{V}^{n-r}$ corresponding to S_{n-r-1} is *conjugate*, in Walker's sense, to $\mathbf{V}^r$. Inasmuch as S_{r-1} and S_{n-r-1}, and hence $\mathbf{V}^r$ and $\mathbf{V}^{n-r}$, have the same null part, the s null vectors in a normal basis for $\mathbf{V}^{n-r}$ may be taken the same as in any given normal basis for $\mathbf{V}^r$. Every vector of $\mathbf{V}^{n-r}$ is orthogonal to every vector of $\mathbf{V}^r$.

The $(n-r)$-plane conjugate to a parallel r-plane is also parallel [Walker, loc. cit., Theorem 3.1]. This all but obvious fact is easily proved analytically by use of the recurrence-formulae (2.1).

A V_n necessarily admits a parallel n-plane, as pointed out by Walker, the n-plane at any point being the whole tangent vector-space at that point. Its conjugate is the 0-plane.

The *intersection* of two planes, $\mathbf{V}^p$, $\mathbf{V}^q$ say, at a point P of V_n is the plane consisting of the set of all vectors common to them. When the intersection is the 0-plane, I shall say that the planes are *non-intersecting*, because the corresponding linear spaces S_{p-1}, S_{q-1} in S_{n-1} do not intersect. It may here be noted that the 0-plane of V_n, and it

* Bertini, *Geometria proiettiva degli iperspazi* (2nd ed., 1923), 147.

† This use of *nullity* is different from that of Y. C. Wong, *Annals of Math.* 46 (1945) 158, who uses the word to mean half the number of null vectors in a 'quasi-orthogonal ennuple'.

alone of all the vector spaces at a point of V_n, has no counterpart in S_{n-1}.

3. Parallel planes in a V_4

A V_4 may admit

(i) (*a*) *A non-null parallel 1-plane.* Such a 1-plane, if it exists, is defined by a non-null vector field h^i, which may be taken to be unit, such that

$$h^i_{,j} = A_j h^i.$$

Multiplication by h^i at once shows that $A_j = 0$, so that the space in fact admits a parallel vector field. Therefore there exist coordinate-systems (x^i) such that ds^2 has the form*

$$ds^2 = \pm(dx^1)^2 + \sum_{2}^{4} g_{\alpha\beta}(x^2, x^3, x^4)\, dx^\alpha dx^\beta.$$

Thus the V_4 is the product of a V_1 and a V_3.

(*b*) *A null parallel 1-plane.* This possibility is dealt with in § 6 below.

(ii) (*a*) *A non-null parallel 2-plane.* In this case the V_4 is the product of two V_2's; that is, there exist coordinate-systems (x^i) such that ds^2 has the form†

$$ds^2 = \sum_{1,2} g_{\alpha\beta}(x^1, x^2)\, dx^\alpha dx^\beta + \sum_{3,4} g_{\alpha\beta}(x^3, x^4)\, dx^\alpha dx^\beta.$$

(*b*) *A null parallel 2-plane.* This case, in some ways the most fundamental of the partially null cases, is considered first (§ 5).

(*c*) *A semi-null parallel 2-plane* (that is, a 2-plane of nullity 1). This case is examined in § 7.

There are no further distinct cases other than the relatively trivial one of the tangent 4-plane and the conjugate 0-plane, inasmuch as the vector space conjugate to a 3-plane is a 1-plane, so that the theory of parallel 3-planes in V_4 comes under (i).

4. Null quasi-orthogonal ennuple in a V_4

In this section, as a preliminary to the main investigation, I obtain certain formulae which hold in any analytic V_4. The notation (uv) will be used for the inner product $u_i v^i = u^i v_i$ of any two vectors.

* Eisenhart, *Riemannian Geometry*, 71.

† Duschek-Mayer, *Lehrbuch der Differentialgeometrie*, Bd II, 152.

Suppose, in a V_4, we are given two independent orthogonal null vector fields ξ^i, η^i, so that we have

$$(\xi\xi) = 0 = (\eta\eta) = (\xi\eta). \tag{4.1}$$

In the projective S_3 at any given point, ξ^i and η^i are points on the same generator of the fundamental quadric.

Now there is an infinite set of null vector fields ξ'^i such that ξ'^i is perpendicular to η^i but not to ξ^i. Thus

$$(\xi'\xi') = 0 = (\xi'\eta), \tag{4.2}$$

but

$$(\xi'\xi) \neq 0. \tag{4.3}$$

Choose any such vector field. In S_3, ξ'^i represents any point, other than η^i, on the second generator of the fundamental quadric through η^i; ξ'^i being one such point, the others are $\lambda\xi'^i+\mu\eta^i$, λ and μ being scalars.

The vector ξ'^i having been chosen, a fourth null vector field η'^i is determined, save for a scalar factor, such that η'^i is perpendicular to ξ^i and to ξ'^i but not to η^i. So

$$(\eta'\eta') = 0 = (\xi'\eta') = (\xi\eta'), \tag{4.4}$$

but

$$(\eta'\eta) \neq 0. \tag{4.5}$$

In S_3, the point η'^i is the intersection of the generators through ξ^i, ξ'^i not already considered.

Of the null vector fields ξ^i, η^i, ξ'^i, η'^i, each is therefore perpendicular to itself and to two of the others, but not to the third. Equations (4.1), (4.2), (4.4) will be referred to as the *orthogonality conditions*. In S_3, (ξ^i, ξ'^i) and (η^i, η'^i) are pairs of opposite vertices of a skew quadrilateral lying upon the fundamental quadric. The plane $(\xi\eta\eta')$ through ξ^i, η^i, η'^i is the tangent plane at ξ^i and has coordinates ξ_i. Similarly η_i, ξ'_i, η'_i, are the tangent planes $(\eta\xi'\xi)$, $(\xi'\eta'\eta)$, $(\eta'\xi\xi')$ at η^i, ξ'^i, η'^i. It will be noted that no assertion is made about the reality of the four null vectors. They can in fact all be real only when the signature of ds^2 is zero.

By multiplying ξ'^i and η'^i by suitable scalars, we can without loss of generality normalize their components in terms of ξ^i, η^i, so as to make

$$(\xi\xi') = 1, \qquad (\eta\eta') = 1, \tag{4.6}$$

in accordance with (4.3) and (4.5). These will be called the *normalizing conditions*.

Now, if X^i are current coordinates in the S_3 associated with any point of V_4, and if we write

$$X = \xi_i X^i, \qquad Y = \eta_i X^i, \qquad X' = \xi'_i X^i, \qquad Y' = \eta'_i X^i,$$

then (X, Y, X', Y') are new coordinates having $(\xi\eta\xi'\eta')$ as tetrahedron of reference. In terms of them the fundamental quadric has an equation of the form

$$2\alpha XX' + 2\beta YY' = 0,$$

four of the edges of the tetrahedron (for example $X = 0 = Y$) being generators. But this equation is

$$(2\alpha\xi_i\xi'_j + 2\beta\eta_i\eta'_j)X^iX^j = 0,$$

so, inasmuch as the equation of the fundamental quadric in terms of the original coordinates X^i is $g_{ij}X^iX^j = 0$, we have

$$g_{ij} = \alpha(\xi_i\xi'_j + \xi'_i\xi_j) + \beta(\eta_i\eta'_j + \eta'_i\eta_j)$$

for some α, β. From the normalizing conditions (4.6), it at once follows that $\alpha = 1 = \beta$, and so

$$g_{ij} = \xi_i\xi'_j + \xi'_i\xi_j + \eta_i\eta'_j + \eta'_i\eta_j. \tag{4.7}$$

The null vector fields ξ^i, η^i, ξ'^i, η'^i are a special case of a quasi-orthogonal ennuple [Wong, loc. cit.].

Covariant differentiation of the equation $(\xi\xi) = 0$ gives $\xi_i\xi^i_{,j} = 0$. This shows that, *for fixed* j, the point in S_3 of coordinates $\xi^i_{,j}$ lies in the tangent plane ξ_i of the fundamental quadric, that is, in the plane $(\xi\eta\eta')$. Hence there exist A_j, B_j, V'_j such that

$$\xi^i_{,j} = A_j\xi^i + B_j\eta^i + V'_j\eta'^i. \tag{4.8}$$

Similarly, from $(\eta\eta) = 0 = (\xi'\xi') = (\eta'\eta')$ we get

$$\eta^i_{,j} = C_j\xi^i + D_j\eta^i + W'_j\xi'^i, \tag{4.9}$$

$$\xi'^i_{,j} = A'_j\xi'^i + B'_j\eta'^i + V_j\eta^i, \tag{4.10}$$

$$\eta'^i_{,j} = C'_j\xi'^i + D'_j\eta'^i + W_j\xi^i. \tag{4.11}$$

These are recurrence formulae of the type (2.1) for the four null vector fields. From the point of view of the theory of parallel vector spaces, they say no more than that the whole tangent space is parallel.

Multiplying (4.8) by η_i, (4.9) by ξ_i, using the orthogonality and normalizing conditions, and, adding, we at once get $V'_j + W'_j = 0$, or $V'_j = -W'_j$. Similarly, multiplying (4.8) by ξ'_i and (4.10) by ξ_i and

adding, we get $A'_j = -A_j$. Proceeding in this way, we find that (4.8),..., (4.11) become

$$\left.\begin{aligned} \xi^i_{,j} &= A_j \xi^i + B_j \eta^i - W'_j \eta'^i, \quad (a)\\ \eta^i_{,j} &= C_j \xi^i + D_j \eta^i + W'_j \xi'^i, \quad (b)\\ \xi'^i_{,j} &= -A_j \xi'^i - C_j \eta'^i - W_j \eta^i, \quad (c)\\ \eta'^i_{,j} &= -B_j \xi'^i - D_j \eta'^i + W_j \xi^i. \quad (d) \end{aligned}\right\} \tag{4.12}$$

These equations will form the basis of the discussion, given below, of parallel vector spaces in V_4.

By (4.12) and the orthogonality and normalizing conditions, we get

$$W'_j = -\eta_i \xi^i_{,j}. \tag{4.13}$$

Differentiating covariantly, and replacing $\eta_{i,k}$ by the expression for it obtained from (4.12) (b), we get

$$\begin{aligned} W'_{j,k} &= -(C_k \xi_i + D_k \eta_i + W'_k \xi'_i)\xi^i_{,j} - \eta_i \xi^i_{,jk} \\ &= D_k W'_j - W'_k A_j - \eta_i \xi^i_{,jk} \end{aligned}$$

by (4.12) (a) and the orthogonality conditions. Hence, interchanging j, k and subtracting, we obtain

$$W'_{j,k} - W'_{k,j} + D_j W'_k - D_k W'_j + A_j W'_k - A_k W'_j = -R_{hijk} \xi^h \eta^i, \tag{4.14}$$

where R_{hijk} is the curvature tensor. Six relations of this type may be obtained from (4.12), one for each of the coefficients A_j, B_j, C_j, D_j, W'_j, W_j, but this one will be enough for present purposes.

5. Null parallel 2-plane in V_4

The V_4 admits a real null parallel 2-plane if real perpendicular null vector fields ξ^i, η^i exist such that

$$\xi^i_{,j} = A_j \xi^i + B_j \eta^i, \tag{5.1}$$

$$\eta^i_{,j} = C_j \xi^i + D_j \eta^i. \tag{5.2}$$

Completing the quasi-orthogonal ennuple ξ^i, η^i, ξ'^i, η'^i by the introduction of null vector fields ξ'^i, η'^i, as in § 4, we obtain

$$\xi'^i_{,j} = -A_j \xi'^i - C_j \eta'^i - W_j \eta^i, \tag{5.3}$$

$$\eta'^i_{,j} = -B_j \xi'^i - D_j \eta'^i + W_j \xi^i. \tag{5.4}$$

By (4.12) the condition that the V_4 admit a null parallel vector field based on ξ^i, η^i is therefore $W'_j \equiv 0$. From (4.14) we obtain

$$R_{hijk} \xi^h \eta^i = 0. \tag{5.5}$$

Differentiating this covariantly, using (5.1), (5.2), we get

$$R_{hijk,l}\xi^h\eta^i = 0 = R_{hijk,lm}\xi^h\eta^i = R_{hijk,lmn}\xi^h\eta^i = \ldots. \qquad (5.6)$$

These, with (5.5), are integrability conditions for (5.1) and (5.2).

If also $W_j \equiv 0$, then, by (5.3) and (5.4), the V_4 admits another null parallel 2-plane, this time based on ξ'^i, η'^i, the two null 2-planes being non-intersecting. That a non-flat V_4 can admit two such null parallel 2-planes is shown below by an example.

It follows at once from the recurrence-formulae (5.1), (5.2) satisfied by ξ^i, η^i, that the partial differential equations

$$\xi^i\frac{\partial\theta}{\partial x^i} = 0, \qquad \eta^i\frac{\partial\theta}{\partial x^i} = 0$$

form a complete system and therefore have two independent solutions. Let these be $\theta = \bar{x}^3$ and $\theta = \bar{x}^4$, where $\bar{x}^3$, $\bar{x}^4$ are functions of the x's. Let $\theta = \bar{x}^2$ be a solution of the former differential equation taken by itself, independent of $\bar{x}^3$, $\bar{x}^4$, and $\theta = \bar{x}^1$ a similar solution of the latter. Then, if we make the transformation of coordinates $(x^i) \to (\bar{x}^i)$ in V_4, ξ^i and η^i go into $\bar{\xi}^i$, $\bar{\eta}^i$, with all $\bar{\xi}^i$ zero except $\bar{\xi}^1$, and all $\bar{\eta}^i$ zero except $\bar{\eta}^2$. As the recurrence-formulae (5.1), (5.2) are unaltered in form if ξ^i, η^i are replaced by linear combinations (and in particular, by scalar multiples) of themselves, we may take the non-zero components of $\bar{\xi}^i$, $\bar{\eta}^i$ equal to unity. So, dropping the 'bars', we have a coordinate-system (x^i) in which

$$\xi^i = \delta^i_1, \qquad \eta^i = \delta^i_2. \qquad (5.7)$$

Substitution in (5.1), (5.2) gives

$$\begin{Bmatrix} i \\ j\ 1 \end{Bmatrix} = A_j\delta^i_1 + B_j\delta^i_2, \qquad \begin{Bmatrix} i \\ j\ 2 \end{Bmatrix} = C_j\delta^i_1 + D_j\delta^i_2, \qquad (5.8)$$

whence

$$\begin{Bmatrix} 3 \\ j\ 1 \end{Bmatrix} = 0 = \begin{Bmatrix} 4 \\ j\ 1 \end{Bmatrix} = \begin{Bmatrix} 3 \\ j\ 2 \end{Bmatrix} = \begin{Bmatrix} 4 \\ j\ 2 \end{Bmatrix}. \qquad (5.9)$$

Because $(\xi\xi) = 0 = (\eta\eta) = (\xi\eta)$, we obtain, from (5.7),

$$g_{11} = 0 = g_{22} = g_{12}, \qquad (5.10)$$

and hence

$$g^{33} = 0 = g^{44} = g^{34}. \qquad (5.11)$$

Because of (5.10), (5.11), equations (5.9) reduce to two, namely

$$\frac{\partial g_{23}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^2} = 0, \qquad \frac{\partial g_{24}}{\partial x^1} - \frac{\partial g_{14}}{\partial x^2} = 0.$$

From these it follows that there exist functions $f(x^i)$, $\phi(x^i)$ such that

$$g_{13} = \frac{\partial f}{\partial x^1}, \qquad g_{23} = \frac{\partial f}{\partial x^2}; \qquad g_{14} = \frac{\partial \phi}{\partial x^1}, \qquad g_{24} = \frac{\partial \phi}{\partial x^2},$$

and therefore that

$$\left.\begin{aligned} g_{13}\,dx^1+g_{23}\,dx^2 &= df-\frac{\partial f}{\partial x^3}\,dx^3-\frac{\partial f}{\partial x^4}\,dx^4 \\ g_{14}\,dx^1+g_{24}\,dx^2 &= d\phi-\frac{\partial \phi}{\partial x^3}\,dx^3-\frac{\partial \phi}{\partial x^4}\,dx^4 \end{aligned}\right\}. \tag{5.12}$$

But, by (5.10),

$$ds^2 = 2(g_{13}\,dx^1+g_{23}\,dx^2)\,dx^3+2(g_{14}\,dx^1+g_{24}\,dx^2)\,dx^4+ \\ +\sum_{3,4} g_{\alpha\beta}\,dx^\alpha dx^\beta, \tag{5.13}$$

and substitution from (5.12) gives

$$ds^2 = 2\,df dx^3+2\,d\phi dx^4+\sum_{3,4}\gamma_{\alpha\beta}\,dx^\alpha dx^\beta,$$

where the $\gamma_{\alpha\beta}$ depend on the $g_{\alpha\beta}$ and the derivatives of f, ϕ with respect to x^3, x^4. Therefore, if we write x, y, z, t for f, ϕ, x^3, x^4, and α, β, γ for γ_{11}, γ_{22}, γ_{12}, we have the metric in the form*

$$ds^2 = 2\,dxdz+2\,dydt+\alpha\,dz^2+2\gamma\,dzdt+\beta\,dt^2. \tag{5.14}$$

Thus: *If a V_4 admit a real null parallel 2-plane, its metric is reducible to the form* (5.14), *where α, β, γ are functions of x, y, z, t.*

In the (x,y,z,t)-coordinate-system, the basic null vectors ξ^i, η^i with which we started are no longer simply δ^i_1, δ^i_2, but, as is easy to see, are linear combinations of these. There is no loss of generality, however, in taking them still to be δ^i_1, δ^i_2, for this merely means replacing the original ξ^i, η^i by another pair linearly dependent upon them, and making the appropriate changes in the coefficients $A_j,\ldots,D_j$ of the recurrence-formulae.

For the metric in the form (5.14) we have

$$g_{ij} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & \alpha & \gamma \\ 0 & 1 & \gamma & \beta \end{pmatrix}, \qquad g^{ij} = \begin{pmatrix} -\alpha & -\gamma & 1 & 0 \\ -\gamma & -\beta & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{5.15}$$

* (5.14) is a special case of a general canonical form found by A. G. Walker for the metric of a V_n admitting a parallel field of partially null r-planes. His work is not yet published, and I am greatly indebted to him for communicating the result to me privately. I am grateful to him, too, for allowing me to see in manuscript the paper, frequently quoted above, upon which the present work is based.

By (5.8)

$$A_j = \begin{Bmatrix} 1 \\ j\ 1 \end{Bmatrix}, \qquad B_j = \begin{Bmatrix} 2 \\ j\ 1 \end{Bmatrix},$$

$$C_j = \begin{Bmatrix} 1 \\ j\ 2 \end{Bmatrix}, \qquad D_j = \begin{Bmatrix} 2 \\ j\ 2 \end{Bmatrix},$$

from which, on calculating the Christoffel symbols, we obtain

$$\left.\begin{aligned} A_j &= 0,\ 0,\ \tfrac{1}{2}\alpha_x,\ \tfrac{1}{2}\gamma_x \\ B_j &= 0,\ 0,\ \tfrac{1}{2}\gamma_x,\ \tfrac{1}{2}\beta_x \\ C_j &= 0,\ 0,\ \tfrac{1}{2}\alpha_y,\ \tfrac{1}{2}\gamma_y \\ D_j &= 0,\ 0,\ \tfrac{1}{2}\gamma_y,\ \tfrac{1}{2}\beta_y \end{aligned}\right\}, \tag{5.16}$$

the suffixes x, y denoting partial derivatives.

We already have

$$\left.\begin{aligned} \xi^i &= 1,\ 0,\ 0,\ 0 = \delta^i_1 \\ \eta^i &= 0,\ 1,\ 0,\ 0 = \delta^i_2 \end{aligned}\right\}, \tag{5.17}$$

and hence, by (5.15),

$$\left.\begin{aligned} \xi_i &= 0,\ 0,\ 1,\ 0 = \delta^3_i \\ \eta_i &= 0,\ 0,\ 0,\ 1 = \delta^4_i \end{aligned}\right\}. \tag{5.18}$$

The null vectors ξ'^i, η'^i of (5.3), (5.4) are auxiliaries. Unless $W_j \equiv 0$, in which case there is the second null parallel 2-plane already referred to, they are in themselves of no immediate importance. It will, however, be useful to obtain formulae for them.

Because $\xi_i \xi'^i = 1$ and $\eta_i \xi'^i = 0$, by (4.6) and (4.2), we have, by (5.18), $\xi'^3 = 1$, $\xi'^4 = 0$; and because $\xi^i \xi'_i = 1$ and $\eta^i \xi'_i = 0$, we have $\xi'_1 = 1$, $\xi'_2 = 0$, by (5.17). Proceeding in this way, using all the orthogonality and normalizing conditions (4.1), (4.2), (4.4), and (4.6), it is easy to show that ξ'^i, η'^i, ξ'_i, η'_i have the forms

$$\left.\begin{aligned} \xi'^i &= -\tfrac{1}{2}\alpha,\quad r,\quad 1,\quad 0 \\ \eta'^i &= -(\gamma+r),\ -\tfrac{1}{2}\beta,\quad 0,\quad 1 \end{aligned}\right\}, \tag{5.19}$$

$$\left.\begin{aligned} \xi'_i &= 1,\quad 0,\quad \tfrac{1}{2}\alpha,\quad \gamma+r \\ \eta'_i &= 0,\quad 1,\quad -r,\quad \tfrac{1}{2}\beta \end{aligned}\right\}, \tag{5.20}$$

where r is an arbitrary function of x, y, z, t and α, β, γ are the coefficients appearing in the metric. The appearance of an arbitrary function was to be expected, because, as seen in § 4, there is an infinite number of null vector fields satisfying the required orthogonality conditions. In order that ξ'^i, η'^i shall denote definite vector fields, it must be supposed that a definite r has been chosen, for example $r = 0$ or $r = -\frac{1}{2}\gamma$. From (5.20), (5.18), and (5.15) it is easy to verify (4.7).

From (5.3) we obtain

$$\begin{aligned} W_j &= -\eta_i' \xi'^i \\ &= -\xi'^2_{,j} + r\xi'^3_{,j} - \tfrac{1}{2}\beta \xi'^4_{,j} \end{aligned}$$

by (5.20). From (5.19) we then obtain, after some calculation,

$$\left.\begin{aligned} W_1 = -\tfrac{1}{2}(\gamma_x + 2r_x), \qquad W_2 &= -\tfrac{1}{2}(\gamma_y + 2r_y) \\ W_3 = -(\gamma_z + r_z) + \tfrac{1}{4}\alpha\gamma_x - \tfrac{1}{2}(\gamma + r)\alpha_x - \tfrac{1}{2}r\gamma_y - \tfrac{1}{4}\beta\alpha_y + \tfrac{1}{2}\alpha_t \\ W_4 = -r_t - \tfrac{1}{4}\beta\gamma_y - \tfrac{1}{2}r\beta_y - \tfrac{1}{2}(\gamma + r)\gamma_x + \tfrac{1}{4}\alpha\beta_x - \tfrac{1}{2}\beta_z \end{aligned}\right\}. \quad (5.21)$$

This is all that need be said for the time being about a V_4 admitting a single null parallel 2-plane, except for the remark that, if α, β, γ in (5.14) happen to be functions of z, t only, then the ds^2 is that of a simply harmonic space of recurrent curvature.* The V_4 then admits two null parallel vector fields ξ^i, η^i, and the parallel 2-plane for which they are a basis is, in Walker's terminology, strictly parallel.

Suppose now that the V_4 admits *two* non-intersecting null parallel 2-planes. Then we must suppose that we are given two independent orthogonal null vector fields ξ^i, η^i as a basis for the one, satisfying (5.1), (5.2), namely

$$\xi^i_{,j} = A_j \xi^i + B_j \eta^i, \qquad \eta^i_{,j} = C_j \xi^i + D_j \eta^i, \quad (5.22)$$

and also two other null vector fields ξ'^i, η'^i which are independent of one another and of ξ^i, η^i, and which satisfy recurrence relations of the form

$$\xi'^i_{,j} = A'_j \xi'^i + B'_j \eta'^i, \qquad \eta'^i_{,j} = C'_j \xi'^i + D'_j \eta'^i, \quad (5.23)$$

as a basis for the other. Thus ξ'^i and η'^i no longer play the auxiliary role they have had hitherto, but are part of the data.

In S_3, ξ^i and η^i represent two points, which we may suppose definitely selected, on a generator of the fundamental quadric, and ξ'^i, η'^i two points on another generator of the same system. Inasmuch as ξ'^i, η'^i may be any two points on the latter generator, the recurrence formulae (5.23) being unaltered in form by the replacement of ξ'^i, η'^i by two other null vectors linearly dependent upon them, we may suppose that ξ'^i, η'^i are the points where the generators of the other system through ξ^i, η^i meet the generator $(\xi'\eta')$. In other words, there is no loss of generality in supposing that the points ξ^i, η^i, ξ'^i, η'^i are the vertices of a skew quadrilateral on the fundamental quadric, even though ξ'^i, η'^i are initially supposed unrelated to ξ^i, η^i.

* Ruse, *Proc. London Math. Soc.* 50 (1949), 327.

Hence the case of two null parallel 2-planes can be brought under the preceding analysis merely by taking $W_j = 0$ in the recurrence-formulae (5.3), (5.4), so that the formulae appropriate to the present case are (5.22) and, instead of (5.23),

$$\xi^i_{,j} = -A_j\xi^i - C_j\eta^i, \qquad \eta'^i_{,j} = -B_j\xi^i - D_j\eta'^i. \tag{5.24}$$

I have not sought a general canonical form for the metric of a V_4 admitting two non-intersecting null parallel 2-planes. The metric could initially be taken in the form (5.14), but the coefficients α, β, γ would then have to satisfy the equations $W_j = 0$, which, as will be seen from (5.21), are complicated. In those equations r must be regarded as a given function of the coordinates.

Suppose $r = 0$. Then a particular solution of the equations $W_j = 0$ is, by (5.21), clearly obtained by taking

$$\beta = 0 = \gamma, \qquad \alpha = \alpha(x, y, z),$$

since, with r, β, γ all zero, the equations reduce to $\alpha_4 = 0$. The metric is now

$$ds^2 = 2\,dxdz + 2\,dydt + \alpha(x, y, z)\,dz^2, \tag{5.25}$$

and by (5.16) we have

$$A_j = \tfrac{1}{2}\alpha_x\delta^3_j, \qquad B_j = 0, \qquad C_j = \tfrac{1}{2}\alpha_y\delta^3_j, \qquad D_j = 0.$$

The recurrence-formulae for this space are therefore

$$\xi^i_{,j} = A_j\xi^i, \qquad \eta^i_{,j} = C_j\xi^i,$$
$$\xi'^i_{,j} = -A_j\xi'^i - C_j\eta'^i, \qquad \eta'^i_{,j} = 0.$$

Thus the parallel 2-plane $(\xi\eta)$ contains a parallel 1-plane (based on ξ^i), and the parallel 2-plane $(\xi'\eta')$ contains a parallel null vector field η'^i. It may be noted that ξ^i and η'^i are also the basis of a parallel 2-plane, and that ξ^i, η^i, η'^i are together a non-normal basis of a parallel 3-plane.

Very special though this example is, it does establish the existence of non-flat spaces V_4 that admit two non-intersecting null parallel 2-planes.

6. Null parallel 1-plane in V_4

To say that a V_4 admits a null parallel 1-plane means that there exists a null vector field ξ^i such that

$$\xi^i_{,j} = A_j\xi^i. \tag{6.1}$$

The 1-plane is the totality of null vectors $\omega\xi^i$ (ω scalar) co-directional with ξ^i.

In the present case we are given no other null vector fields, but we may choose three others, η^i, ξ'^i, η'^i, to form with ξ^i a null quasi-orthogonal ennuple, as in § 4. The recurrence-formulae (4.12), with $B_j = 0 = W_j'$, then give, along with (6.1),

$$\eta^i_{,j} = C_j \xi^i + D_j \eta^i, \tag{6.2}$$

$$\xi'^i_{,j} = -A_j \xi'^i - C_j \eta'^i - W_j \eta^i, \tag{6.3}$$

$$\eta'^i_{,j} = -D_j \eta'^i + W_j \xi^i. \tag{6.4}$$

Equations (6.1) and (6.2) show that ξ^i, η^i are basis-vectors of a null parallel 2-plane, and (6.1), (6.4) that ξ^i, η'^i are basis-vectors of another such 2-plane. The two 2-planes intersect in the given 1-plane $\omega\xi^i$. Equation (6.3) is auxiliary, and is of no immediate interest.

Further, the three equations (6.1), (6.2), (6.4) taken together define a parallel 3-plane for which the vector fields ξ^i, η^i, η'^i form a non-normal basis. This is the 3-plane conjugate to the given parallel 1-plane $\omega\xi^i$. Its counterpart in the projective S_3 at any point of V_4 is the tangent plane at ξ^i to the fundamental quadric.

A normal basis for the conjugate 3-plane is obtained by taking, with ξ^i, the vectors h^i, k^i defined by

$$\left.\begin{aligned} h^i &= p\eta^i + q\eta'^i \\ k^i &= p\eta^i - q\eta'^i \end{aligned}\right\}, \tag{6.5}$$

where p, q are any scalars (or, in particular, constants) such that $pq = \frac{1}{2}$. Because $(\eta\eta) = 0 = (\eta'\eta')$ and $(\eta\eta') = 1$, by (4.1), (4.4), and (4.6), we see at once that

$$(hh) = 1, \qquad (kk) = -1, \qquad (hk) = 0. \tag{6.6}$$

Hence ξ^i, h^i, k^i form a normal basis, ξ^i being a null vector and h^i, k^i unit. In S_3, h^i and k^i represent points in the tangent plane ξ_i of the fundamental quadric, the lines (ξh) and (ξk) being harmonically conjugate with respect to the generators $(\xi\eta)$ and $(\xi\eta')$ through ξ^i. The points h^i and k^i are conjugate with respect to the quadric.

The recurrence-formulae (6.1),..., (6.4) for the present case are the same as the corresponding formulae (5.1),..., (5.4) of § 5, with $B_j = 0$. Hence the metric is in the present case reducible to the form (5.14), the condition $B_j = 0$ giving, by (5.16),

$$\beta = \beta(y, z, t), \qquad \gamma = \gamma(y, z, t).$$

Summarizing the results of this section, we therefore have: *If a V_4 admit a parallel 1-plane, it also admits two null parallel 2-planes intersecting in the 1-plane. These planes are all contained in the parallel 3-plane conjugate to the given 1-plane. When the parallel planes are all real, the metric of the V_4 is reducible to the form*

$$ds^2 = 2\,dxdz+2\,dydt+\alpha(x,y,z,t)\,dz^2+ \\ +2\gamma(y,z,t)\,dzdt+\beta(y,z,t)\,dt^2. \quad (6.7)$$

The example (5.25) given at the end of the last section is a particular case of this, as well as being an example of a V_4 admitting two non-intersecting null parallel 2-planes.

7. Semi-null parallel 2-plane in V_4

We now suppose that a V_4 admits a parallel 2-plane having as basis two perpendicular vectors ξ^i, h^i, of which the former is null and the latter unit. Thus we now assume

$$\xi^i_{,j} = A_j\xi^i+E_jh^i, \quad (7.1)$$

$$h^i_{,j} = H_j\xi^i+F_jh^i, \quad (7.2)$$

where $(\xi\xi) = 0, \quad (hh) = e_1, \quad (\xi h) = 0 \quad (e_1 = \pm 1).$ (7.3)

In S_3, ξ^i represents a point on the fundamental quadric and h^i any point on a line touching the quadric at ξ^i.

Multiply (7.1) by h_i, (7.2) by ξ_i, and add. Then by (7.3) we at once get $E_j = 0$. Now multiply (7.2) by h_i, again using (7.3), and we get $F_j = 0$. Hence

$$\xi^i_{,j} = A_j\xi^i, \quad (7.4)$$

$$h^i_{,j} = H_j\xi^i. \quad (7.5)$$

Thus, by (7.4), *the V_4 admits a null parallel 1-plane contained in the given semi-null parallel 2-plane.** It therefore also admits the parallel 3-plane conjugate to the 1-plane $\omega\xi^i$.

Now let k^i be a unit vector field perpendicular to both ξ^i and h^i, so that

$$(kk) = e_2, \quad (\xi k) = 0, \quad (hk) = 0 \quad (e_2 = \pm 1). \quad (7.6)$$

In S_3, k^i is any point on the tangent of the fundamental quadric polar to the tangent joining ξ^i and h^i. Both tangents touch the quadric at ξ^i, and lie in the tangent plane ξ_i. The points h^i, k^i are conjugate.

Because $h_ik^i = 0$ and $\xi_ik^i = 0$, we get, on differentiating the former equation covariantly and using (7.5),

$$h_ik^i_{,j} = 0.$$

* Cf. Walker, loc. cit., Theorem 3.1 (ii).

Hence, *for fixed* j, $k^i_{,j}$ represents in S_3 a point in the polar plane h_i of the point h^i. This polar plane passes through the tangent (ξk) because the points h^i, k^i are conjugate. Similarly, differentiating $\xi_i k^i = 0$ covariantly and using (7.4), we find that $\xi_i k^i_{,j} = 0$. Therefore the point $k^i_{,j}$ (j fixed) also lies in the tangent plane ξ_i. It therefore lies upon the tangent (ξk), and so

$$k^i_{,j} = K_j \xi^i + G_j k^i$$

for some K_j, G_j. Multiplying this by k_i and using (7.6), we get $G_j = 0$, and therefore

$$k^i_{,j} = K_j \xi^i. \tag{7.7}$$

Equations (7.4) and (7.7) show that ξ^i, k^i are the basis of a second semi-null parallel 2-plane. It is the parallel 2-plane conjugate to the one with which we started. Further, if

$$l^i = ah^i + bk^i, \tag{7.8}$$

where a, b are constants such that

$$e_1 a^2 + e_2 b^2 = \pm 1, \tag{7.9}$$

then l^i is a unit vector, and we obtain, from (7.5) and (7.7),

$$l^i_{,j} = L_j \xi^i,$$

where

$$L_j = aH_j + bK_j.$$

This, with (7.4), shows that ξ^i and l^i are also the basis of a semi-null parallel 2-plane. Inasmuch as a, b are any constants satisfying (7.9), it follows that *the* V_4 *admits a pencil of semi-null parallel 2-planes, all intersecting in the parallel 1-plane* $\omega\xi^i$.

In (7.8), instead of choosing a, b to make l^i unit, we may choose them to make l^i null. This requires $e_1 a^2 + e_2 b^2 = 0$, and so, if the null vectors obtained in this way are to be real, e_1 and e_2 must have opposite signs, say $e_1 = 1$, $e_2 = -1$. Then $b = \pm a$. As we are at liberty to choose the factors of proportionality of the null vectors at will, we may take $a = 1/\sqrt{2}$. Denoting the null vectors thus obtained by η^i, η'^i, we then have

$$\eta^i = \frac{1}{\sqrt{2}}(h^i + k^i), \qquad \eta'^i = \frac{1}{\sqrt{2}}(h^i - k^i), \tag{7.10}$$

and, because $(hh) = e_1 = 1$, $(kk) = e_2 = -1$, we also have $(\eta\eta') = 1$.

By (7.5), (7.7), and (7.10) we get

$$\eta^i_{,j} = C_j \xi^i, \tag{7.11}$$

$$\eta'^i_{,j} = W_j \xi^i, \tag{7.12}$$

where

$$C_j = \frac{1}{\sqrt{2}}(H_j + K_j), \qquad W_j = \frac{1}{\sqrt{2}}(H_j - K_j).$$

The pairs of equations (7.4), (7.11) and (7.4), (7.12) show that *the V_4 admits two null parallel 2-planes, which also intersect in the 1-plane $\omega\xi^i$.* In itself, this result was to be expected from the work of § 6, in which it was shown that a V_4 admitting a null parallel 1-plane also admits two null parallel 2-planes. What was not to be foreseen was the special form of (7.11) and (7.12), which involve only ξ^i on the right-hand side.

The last two recurrence formulae, (7.11) and (7.12), were obtained by combining (7.5) and (7.7) linearly. Hence (7.5) and (7.7) may be obtained, conversely, from (7.11) and (7.12). The space is therefore fully characterized by the three equations

$$\xi^i_{,j} = A_j\xi^i, \qquad \eta^i_{,j} = C_j\xi^i, \qquad \eta'^i_{,j} = W_j\xi^i,$$

and these are the same as the corresponding formulae in § 5, namely (5.1), (5.2), and (5.4), if in the latter we put $B_j = 0 = D_j$. This, by (5.16), gives

$$\beta_x = 0 = \beta_y = \gamma_x = \gamma_y,$$

and β and γ are therefore functions of z, t only. Hence: *If a V_4 admit a semi-null parallel 2-plane, it also admits a null 1-plane, a pencil of semi-null 2-planes intersecting in the 1-plane, and two null 2-planes which also intersect in the 1-plane. These parallel planes are all contained in the parallel 3-plane conjugate to the null 1-plane. When the parallel planes are all real, the metric of the V_4 is reducible to*

$$ds^2 = 2\,dxdz + 2\,dydt + \alpha(x,y,z,t)\,dz^2 + 2\gamma(z,t)\,dzdt + \beta(z,t)\,dt^2. \tag{7.13}$$

The particular metric (5.25), which has already twice been used as an example, also provides an example of the case at present under discussion. The space (5.25) in fact exhibits simultaneously all the features discussed in detail in the present paper: it admits a null parallel 1-plane, two non-intersecting null parallel 2-planes, two intersecting null parallel 2-planes, a pencil of semi-null 2-planes, and a parallel 3-plane of nullity 1.

8. Remark on null parallel planes in any V_n

The fact that the existence of a null parallel 1-plane or of a semi-null parallel 2-plane in a V_4 implies the existence of parallel planes other than the null parts or conjugates of the given planes, as shown in §§ 6, 7, was hardly to be foreseen. In both cases the 'latent' parallel planes appeared as adjuncts of a null 1-plane.

This might suggest that a null parallel 1-plane has some special generative quality in any V_n, in the sense that its existence automatically requires the existence of parallel planes other than its conjugate. A consideration of the general case quickly shows, however, that the corresponding property is held in a V_n of even dimensionality, when $n > 4$, not by a null parallel 1-plane, but by a null parallel $(\frac{1}{2}n-1)$-plane. An account of these, and of similar matters affecting spaces of odd dimensionality, will be given in a later paper.

A NOTE ON POWER SERIES IN MATRICES*

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ACCORDING to Dienes's definition a sequence of matrices $\{S_n\}$ is said to *converge* to S as $n \to \infty$ if all the sequences $\{(S_n)_{i,j}\}$ for every fixed i and j, converge to $S_{i,j}$. The convergence of series follows as usual from considering the sequence of partial sums.

We notice that convergent series in infinite matrices may have one or more of three forms of convergence, namely

(i) *ordinary convergence*, which is usually referred to as 'convergence';

(ii) *absolute convergence*, in which a series $\sum_{p=1}^{\infty} c_p A^{(p)}$ is absolutely convergent if $\sum_{p=1}^{\infty} |c_p||a_{i,j}^{(p)}|$, where $a_{i,j}^{(p)}$ is the (i,j)th element in $A^{(p)}$, is convergent for every i and j; and

(iii) *strong convergence*, in which a series $\sum^{\infty} c_p A^{(p)}$ is strongly convergent if $\sum^{\infty} |c_p||A^{(p)}|$ is convergent, where $|A^{(p)}|$ is the bound of the matrix $A^{(p)}$, the matrix bound being taken according to the following modified definition.

If by a given rule a number $|A| \geqslant 0$ is assigned to every matrix A of a given field $\mathfrak{F}$ satisfying the conditions:

(a) $|cA| = |c||A|$, $|I| = 1$, where c is any scalar and I is the unit matrix;

(b) $|A+B| \leqslant |A|+|B|$;

(c) $|AB| \leqslant |A||B|$; and

(d) $|a_{i,j}| \leqslant |A|$,

provided that A and B are matrices of the given field, then $|A|$ is said to be the *bound* of A.

It is easily seen that a strongly convergent series is absolutely convergent, and an absolutely convergent series is convergent.

The idea of convergence of power series in matrices with bounded associative fields was mentioned in Dienes's paper where he proved

* By 'matrix' I shall always mean 'infinite matrix', and by 'series' I shall mean 'series in infinite matrices'.

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that, if A is a matrix of a field $\mathfrak{F}$ with a semi-closed associative bound and if $|A| < 1$, then* $\sum_{p=0}^{\infty} A^p$ is absolutely convergent.

It is easily seen that the following example provides a case in which the series $\sum_{0}^{\infty} A^p$ converges absolutely although the matrix A does not satisfy the conditions of the theorem.

Take A to be the matrix for which $a_{i,i} = \alpha$, $a_{1,2} = 2\alpha$, where $\frac{1}{2} \leqslant |\alpha| < 1$, all other elements being zero. Then $|A| \not< 1$, although the series $\sum_{0}^{\infty} A^p$ converges absolutely.

To cover such a case and to consider the general power series I give the following extension to Dienes's theorem:

If A is a self-associative matrix, then a sufficient condition for the series $\sum_{n=0}^{\infty} c_n A^n$ to converge absolutely is that there exist co-prime positive integers r and s such that A^r and A^s are two matrices of a bounded associative field $\mathfrak{F}$ and in addition $|A^r| < \rho^r$ (or $|A^s| < \rho^s$), where ρ is the radius of convergence of $\sum_{n=0}^{\infty} c_n z^n$. (By *self-associative matrix* I mean a matrix which can be a member of some associative field.)

Proof. Let n be an integer not less than rs. Then n can be written in the form

$$n = lr + ms,$$

where l and m are integers such that $l > 0$ and $0 \leqslant m < r$.

Hence
$$A^n = A^{lr+ms}$$
$$= (A^r)^l \times (A^s)^m,$$

by the self-associative property of A, and therefore A^n belongs to the same field $\mathfrak{F}$.

Applying the definition of the matrix bound we have

$$|A^n| \leqslant |(A^r)^l| \,.\, |(A^s)^m|$$
$$\leqslant (|A^r|)^l \,.\, (|A^s|)^m$$
$$\leqslant \rho_1^{rl} \times c_1^m, \text{ where } |A^r| = \rho_1^r \ (0 \leqslant \rho_1 < \rho) \text{ and } |A^s| = c_1 < \infty,$$
$$\leqslant \rho_1^n \,.\, c,$$

* A matrix raised to the power zero under the sum sign denotes, by convention, the unit matrix.

where
$$c = \begin{cases} 1 & \text{if } \frac{c_1}{\rho_1^s} \leqslant 1, \\ \left(\frac{c_1}{\rho_1^s}\right)^{r-1} & \text{if } \frac{c_1}{\rho_1^s} > 1. \end{cases}$$

Now the series
$$\sum_{n=rs}^{\infty} |c_n||A^n| \leqslant c \sum_{n=rs}^{\infty} |c_n|\rho_1^n,$$

and is therefore convergent.

Since $\sum_{n=rs}^{\infty} c_n A^n$ is strongly convergent, the given series is absolutely convergent.

REFERENCES

1. R. G. Cooke, *Infinite Matrices and Sequence Spaces* (in the press; Macmillan).
2. P. Dienes, 'Notes on linear equations in infinite matrices', *Quart. J. of Math.* (Oxford), 3 (1932), 253–68.
3. R. H. Makar, Ph.D. Thesis, Birkbeck College, University of London (1948).

EIGENFUNCTION EXPANSIONS FOR A FINITE TWO-DIMENSIONAL REGION

By E. C. TITCHMARSH (*Oxford*)

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1. LET D be a region of the (x, y)-plane, of a type to be specified more precisely later. Let $q(x, y)$ be a given function of x and y, defined in D, and consider the partial differential equation

$$\nabla^2\phi+\{\lambda-q(x, y)\}\phi = 0. \tag{1.1}$$

The eigenfunction expansion problem associated with these data consists in proving that an arbitrary function $f(x, y)$ can, subject to certain conditions, be expanded in a series of solutions of (1.1), known as *eigenfunctions*, which vanish round the boundary of D, and which correspond to certain values λ_n of the parameter λ, known as *eigenvalues*. In cases where the region D extends to infinity, the eigenvalues may coalesce into continuous stretches, and the corresponding expansion formulae then involve integrals instead of series.

In a previous paper† I obtained the eigenfunction expansion associated with the partial differential equation (1.1) and the whole (x, y)-plane. This was done in the following stages:

(i) Let the region D be a square and let $q(x, y) = 0$. The formulae are then those of double Fourier series.

(ii) Let the region D be a square, and let $q(x, y)$ be any function with continuous partial derivatives of the first order. This case is obtained from the previous one by a process of iteration.

(iii) The case of an infinite region is then obtained by making the side of the square in case (ii) tend to infinity.

In the present paper the expansion for a fairly general type of finite region will be obtained. We assume case (ii), and make $q(x, y) \to \infty$ outside a region D included in the square. We thus obtain the expansion for functions defined over D and vanishing round the boundary of D.

2. Let D be a region included in a simple closed curve. We shall suppose that this curve consists of a finite number of pieces, which

† Communicated to the London Mathematical Society. The known theory of these expansions is given by Courant and Hilbert, *Methoden der Math. Physik* (ii), Kaps. 4 and 7.

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are either of the form $y = F(x)$, where $F(x)$ is continuous and $\{F(x+h)-F(x)\}/h$ is bounded, or are of a similar form with x and y interchanged. Two such pieces of curve may be joined at an angle, but not so as to form a cusp. We shall suppose that D includes the a-square

$$-a \leqslant x \leqslant a, \qquad -a \leqslant y \leqslant a,$$

and is included in the b-square. Let $q(x, y)$ have continuous partial derivatives of the first order in D (the boundary included).

If (x, y) is outside D, let $r(x, y)$ denote the distance of (x, y) from D. If (x, y) is inside or on the boundary of D, let $r(x, y) = 0$. Then, as $\nu \to \infty$, $q(x, y)+\nu r(x, y)$ tends steadily to infinity outside D. This function, however, may not have continuous partial derivatives. But, since we can approximate uniformly to any continuous function by polynomials,† we can, for each ν, define a polynomial $q^{(\nu)}(x, y)$ which differs from

$$q(x, y)+\nu r(x, y)-1/\nu$$

by at most $1/\{2\nu(\nu+1)\}$. Then $q^{(\nu)}(x, y)$ has continuous partial derivatives of all orders, $q^{(\nu+1)}(x, y) \geqslant q^{(\nu)}(x, y)$, and $q^{(\nu)}(x, y)$ tends uniformly to $q(x, y)$ inside D, and to infinity outside D.

Let $\lambda_n^{(\nu)}$, $\psi_n^{(\nu)}(x, y)$ be the eigenvalues and eigenfunctions for the b-square and the differential equation involving $q^{(\nu)}(x, y)$, with the boundary condition $\psi_n^{(\nu)}(x, y) = 0$ on the square; and μ_n the eigenvalues for the a-square and $q(x, y)$. The μ_n are thus independent of ν.

For any function $f(x, y)$, let

$$J^{(\nu)}(f) = \int_{-b}^{b}\int_{-b}^{b} (f_x^2+f_y^2+q^{(\nu)}f^2)\, dxdy,$$

where suffixes denote partial differentiations. Then

$$J^{(\nu)}(\psi_n^{(\nu)}) = \int_{-b}^{b}\int_{-b}^{b}\left(-\psi_n^{(\nu)}\frac{\partial^2\psi_n^{(\nu)}}{\partial x^2}-\psi_n^{(\nu)}\frac{\partial^2\psi_n^{(\nu)}}{\partial y^2}+q^{(\nu)}\psi_n^{(\nu)2}\right) dxdy,$$

by integration by parts, the integrated terms vanishing. Now

$$\nabla^2\psi_n^{(\nu)}+\{\lambda_n^{(\nu)}-q^{(\nu)}(x, y)\}\psi_n^{(\nu)} = 0. \tag{2.1}$$

Hence

$$J^{(\nu)}(\psi_n^{(\nu)}) = \lambda_n^{(\nu)}\int_{-b}^{b}\int_{-b}^{b}\psi_n^{(\nu)2}\, dxdy = \lambda_n^{(\nu)}. \tag{2.2}$$

† See e.g. Hobson, *Theory of Functions of a Real Variable* (ii) (2nd ed.), § 160.

Now each $\lambda_n^{(\nu)}$ increases with $q^{(\nu)}$, and also, for a given $q^{(\nu)}$, increases as the square to which the expansion applies decreases.† Thus, for each n,

$$\lambda_n^{(\nu)} \leqslant \lambda_n^{(\nu+1)} \leqslant \ldots \leqslant \mu_n.$$

Hence each $\lambda_n^{(\nu)}$ tends to a finite limit, say λ_n, as $\nu \to \infty$. Also

$$J^{(\nu)}(\psi_n^{(\nu)}) \leqslant \mu_n. \tag{2.3}$$

3. It follows from (2.3) that

$$\int_{-b}^{b}\int_{-b}^{b} (\psi_{n,x}^{(\nu)2}+\psi_{n,y}^{(\nu)2})\,dxdy \leqslant \mu_n - m, \tag{3.1}$$

where m is the lower bound of all the functions $q^{(\nu)}$. Hence, by a theorem of Rellich,‡ there is a sequence of values of ν through which $\psi_n^{(\nu)}$ converges in mean; i.e. there is a function $\psi_n(x, y)$ such that

$$\int_{-b}^{b}\int_{-b}^{b} (\psi_n - \psi_n^{(\nu)})^2\,dxdy \to 0 \tag{3.2}$$

as $\nu \to \infty$ through the sequence.

Now, if (x_0, y_0) is any point in the b-square, distant at least R from the perimeter,§ and

$$r = \sqrt{\{(x-x_0)^2+(y-y_0)^2\}},$$

then

$$\psi_n^{(\nu)}(x_0, y_0) = \frac{1}{\pi R^2}\iint_{r<R} \psi_n^{(\nu)}\,dxdy + \\ + \frac{1}{2\pi}\iint_{r<R} \left\{\frac{1}{2}\left(1-\frac{r^2}{R^2}\right) - \log\frac{R}{r}\right\}(q^{(\nu)}-\lambda_n^{(\nu)})\psi_n^{(\nu)}\,dxdy. \tag{3.3}$$

If (x_0, y_0) is in D, and R is small enough, then, as $\nu \to \infty$ through the sequence of values involved in (3.2), the right-hand side tends to the corresponding expression without (ν). Hence $\psi_n^{(\nu)}$ also tends to this limit, and indeed uniformly in any region entirely inside D. Thus

$$\psi_n(x_0, y_0) = \frac{1}{\pi R^2}\iint_{r<R} \psi_n\,dxdy + \\ + \frac{1}{2\pi}\iint_{r<R} \left\{\frac{1}{2}\left(1-\frac{r^2}{R^2}\right) - \log\frac{R}{r}\right\}(q-\lambda_n)\psi_n\,dxdy. \tag{3.4}$$

† Courant and Hilbert, loc. cit. (i) (2nd ed.), Kap. VI, § 2, Satz 3, Satz 7.
‡ Ibid. ii. 489. § Ibid. ii. 250 (7).

Making $R \to 0$, it follows† that

$$\nabla^2\psi_n + \{\lambda_n - q(x,y)\}\psi_n = 0 \tag{3.5}$$

inside D.

The formulae obtained by differentiating (3.3) and (3.4) with respect to x_0 or y_0 also hold, and it follows that the partial derivatives $\psi^{(\nu)}_{n,x}$ and $\psi^{(\nu)}_{n,y}$ tend to $\psi_{n,x}$ and $\psi_{n,y}$ inside D. It therefore follows from (3.1) and Fatou's theorem that

$$\iint_D (\psi^2_{n,x} + \psi^2_{n,y})\, dxdy \leqslant \mu_n - m. \tag{3.6}$$

Again, if E is a region in the b-square entirely outside D, then $q^{(\nu)} \to \infty$ uniformly in E. Now, by (2.3),

$$\iint_E q^{(\nu)} \psi^{(\nu)2}_n\, dxdy \leqslant \mu_n.$$

Hence

$$\iint_E \psi^{(\nu)2}_n\, dxdy \to 0.$$

Hence $\psi_n = 0$ almost everywhere outside D. Since it is so far only defined almost everywhere, we can take $\psi_n = 0$ everywhere outside D.

We also have

$$\iint_D \psi_m \psi_n\, dxdy = \begin{cases} 1 & (m = n), \\ 0 & (m \neq n), \end{cases} \tag{3.7}$$

as limits of the corresponding relations involving $\psi^{(\nu)}_m$ and $\psi^{(\nu)}_n$.

4. Behaviour of $\psi_n(x,y)$ on the boundary of D

In this section we consider a particular $\psi_n(x,y)$, and suppress the suffix n for convenience in writing. Let $y = F(x)$ $(x_1 \leqslant x \leqslant x_2)$ be part of the lower boundary of D. We have

$$\psi^{(\nu)}(x,y) = \psi^{(\nu)}(x, y-3\delta) + \int_{y-3\delta}^{y} \psi^{(\nu)}_y(x,t)\, dt,$$

where $\delta > 0$. Hence, using an inequality of the form

$$(a+b)^2 \leqslant 2a^2 + 2b^2,$$

we have

$$|\psi^{(\nu)}(x,y)|^2 \leqslant 2|\psi^{(\nu)}(x, y-3\delta)|^2 + 2\left|\int_{y-3\delta}^{y} \psi^{(\nu)}_y(x,t)\, dt\right|^2$$

$$\leqslant 2|\psi^{(\nu)}(x, y-3\delta)|^2 + 6\delta \int_{y-3\delta}^{y} \{\psi^{(\nu)}_y(x,t)\}^2\, dt$$

† Courant and Hilbert, loc. cit. (ii), Kap. IV, §§ 2–3.

by Schwarz's inequality; and so

$$\int_{F(x)+\delta}^{F(x)+2\delta} |\psi^{(\nu)}(x,y)|^2\,dy \leqslant 2\int_{F(x)-2\delta}^{F(x)-\delta} |\psi^{(\nu)}(x,y)|^2\,dy + \\ +6\delta\int_{F(x)+\delta}^{F(x)+2\delta} dy \int_{y-3\delta}^{y} |\psi_y^{(\nu)}(x,t)|^2\,dt.$$

When the order of integration of the last integral is inverted, y varies over a range not exceeding 3δ. Hence this term does not exceed

$$18\delta^2\int_{F(x)-2\delta}^{F(x)+2\delta} |\psi_y^{(\nu)}(x,t)|^2\,dt.$$

Hence

$$\int_{x_1}^{x_2} dx \int_{F(x)+\delta}^{F(x)+2\delta} |\psi^{(\nu)}|^2\,dy \leqslant 2\int_{x_1}^{x_2} dx \int_{F(x)-2\delta}^{F(x)-\delta} |\psi^{(\nu)}|^2\,dy + \\ +18\delta^2\int_{x_1}^{x_2} dx \int_{F(x)-2\delta}^{F(x)+2\delta} |\psi_y^{(\nu)}|^2\,dy.$$

Now make $\nu\to\infty$, keeping δ fixed. The region of integration of the first term on the right may be supposed to lie entirely outside D, and then this term tends to 0. The second integral on the right is bounded, by (3.1). Hence

$$\int_{x_1}^{x_2} dx \int_{F(x)+\delta}^{F(x)+2\delta} |\psi(x,y)|^2\,dy = O(\delta^2). \tag{4.1}$$

Also it follows from (3.6) that

$$\int_{F(x)}^{F(x)+\delta} |\psi_y(x,y)|^2\,dy$$

is finite for almost all valucs of x. For any such x

$$|\psi(x,y)-\psi(x,y')| \leqslant \left(\int_y^{y'} dt \int_y^{y'} |\psi_y|^2\,dt\right)^{\frac{1}{2}} = O\{(y'-y)^{\frac{1}{2}}\}$$

for all y and y' sufficiently near to $F(x)$. Hence $\psi(x,y)$ tends to a limit as $y\to F(x)$, say $\psi(x,y)\to\omega(x)$. Hence

$$\frac{1}{\delta}\int_{F(x)+\delta}^{F(x)+2\delta} |\psi(x,y)|^2\,dy \to \omega^2(x).$$

From this, (4.1), and Fatou's theorem it follows that

$$\int_{x_1}^{x_2} \omega^2(x)\,dx = 0.$$

Hence $\omega(x) = 0$ almost everywhere, i.e. $\psi(x,y) \to 0$ almost everywhere along the boundary.

Now suppose that

$$|F(x+h)-F(x)| \leqslant M|h|$$

on the part of the boundary considered. Then, if $|x-x_0| \leqslant \eta$,

$$F(x)+\delta \leqslant F(x_0)+M\eta+\delta \leqslant F(x_0)+\tfrac{5}{4}\delta$$

if $\eta = \frac{1}{4}\delta/M$; and

$$F(x)+2\delta \geqslant F(x_0)-M\eta+2\delta \geqslant F(x_0)+\tfrac{7}{4}\delta.$$

Hence the circle with centre $(x_0, F(x_0)+\frac{3}{2}\delta)$ and radius $\frac{1}{4}\delta \min(M^{-1}, 1)$ lies between the curves $y = F(x)+\delta$ and $y = F(x)+2\delta$. Denoting this circle by C, and its radius by R, we have

$$\begin{aligned}\frac{1}{R^2}\iint_C |\psi|\,dxdy &\leqslant \frac{1}{R^2}\left\{\iint_C dxdy \iint_C \psi^2\,dxdy\right\}^{\frac{1}{2}} \\ &= O\left[\frac{1}{R}\left\{\iint_C \psi^2\,dxdy\right\}^{\frac{1}{2}}\right] \\ &= O(1)\end{aligned}$$

by (4.1). Hence, by (3.4),

$$\psi(x_0, F(x_0)+\tfrac{3}{2}\delta) = O(1),$$

i.e. ψ is bounded in the neighbourhood of the boundary of D.

A similar argument, allowing for the variation of λ_n, with

$$R = |\lambda_n|^{-\frac{1}{2}},$$

shows that

$$\psi_n = O(|\lambda_n|^{\frac{1}{2}}).$$

5. We have still to prove that ψ tends to zero at the boundary of D. Let (x_0, y_0) be an interior point, at a (small) distance $2R$ from the boundary. Then the circle with centre (x_0, y_0) and radius R lies entirely inside D. Taking the shortest distance from (x_0, y_0) to the boundary as axis of y, let $y = F(x)$ be the equation of the boundary, and $y = K(x)$ that of either the upper or the lower half of the circle. Then, for either $K(x)$,

$$K(x)-F(x) = O(R).$$

Now $$\psi\{x, K(x)\} - \psi\{x, F(x)\} = \int_{F(x)}^{K(x)} \psi_y(x, y)\, dy,$$

and $\psi\{x, F(x)\} = 0$ almost everywhere. Hence

$$\begin{aligned}\int_{-R}^{R} |\psi\{x, K(x)\}|\, dx &\leqslant \int_{-R}^{R} dx \int_{F(x)}^{K(x)} |\psi_y|\, dy \\ &\leqslant \left\{ \int_{-R}^{R} dx \int_{F(x)}^{K(x)} dx \,.\, \int_{-R}^{R} dy \int_{F(x)}^{K(x)} \psi_y^2\, dy \right\}^{\frac{1}{2}} \\ &= \{O(R^2).o(1)\}^{\frac{1}{2}} = o(R).\end{aligned}$$

Taking $x = R\cos\theta$, $y = R\sin\theta$, on the circle, we have

$$dx = -R\sin\theta\, d\theta,$$

so that, writing $\psi(x, y) = \psi[R, \theta]$,

$$\int_{\delta}^{\pi-\delta} |\psi[R, \theta]|\, d\theta = o(\operatorname{cosec}\delta), \qquad \int_{\pi+\delta}^{2\pi-\delta} |\psi[R, \theta]|\, d\theta = o(\operatorname{cosec}\delta).$$

Since ψ is bounded, it follows that

$$\int_{0}^{2\pi} |\psi[R, \theta]|\, d\theta = o(\operatorname{cosec}\delta) + O(\delta) = o(1)$$

by choosing first δ and then R small enough. Now

$$\psi(x_0, y_0) = \frac{1}{2\pi} \int_{0}^{2\pi} \psi[R, \theta]\, d\theta - \frac{1}{2\pi} \iint_{r<R} \left(\log\frac{R}{r}\right)(q-\lambda)\psi\, r dr d\theta,$$

q here being redefined in terms of the new coordinates; and the last term is

$$O\left\{ \iint_{r<R} \left(\log\frac{R}{r}\right) r dr d\theta \right\} = O(R^2).$$

Hence $\psi \to 0$ uniformly round the boundary.

Again, on differentiating (3.4) with respect to x_0, we obtain

$$\begin{aligned}\frac{\partial\psi_n}{\partial x_0} = \frac{1}{\pi R} \int_{0}^{2\pi} \psi_n(x_0 + R\cos\theta, y_0 + R\sin\theta)\cos\theta\, d\theta - \\ - \frac{1}{2\pi} \iint_{r<R} \left(\frac{1}{r} - \frac{r}{R^2}\right)\cos\theta(q-\lambda_n)\psi_n\, dxdy.\end{aligned}$$

Taking (x_0, y_0) at distance ϵ from the boundary, and $R = \frac{1}{2}\epsilon$, it follows that

$$\frac{\partial \psi_n}{\partial x_0} = o\left(\frac{1}{\epsilon}\right).$$

Similarly

$$\frac{\partial \psi_n}{\partial y_0} = o\left(\frac{1}{\epsilon}\right).$$

6. If we impose further conditions on the bounding curves, we can obtain more precise results. Consider a point on the boundary where the curve has a tangent, and where the perpendicular from a neighbouring point of the curve which meets the tangent at distance δ along the tangent is $O(\delta^2)$. Take the point as origin, and the tangent as axis of x. Then, if R is small enough, the circle $x^2+(y-R)^2 = R^2$ lies entirely inside D, except for the origin. Let $y = K(x)$ now denote the circumference of this circle in the neighbourhood of the origin. In this case $F(x) = O(x^2)$, $K(x) = O(x^2)$, and proceeding as before we obtain

$$\int_0^h |\psi\{x, K(x)\}|\, dx = o\,(h^{\frac{3}{2}}).$$

Taking in this case $x = R\sin\theta$, $y = R(1-\cos\theta)$ on the circle, it follows that, for a fixed R and $\delta \to 0$,

$$I(\delta) = \int_0^\delta |\psi[R, \theta]|\, d\theta = o\,(\delta^{\frac{3}{2}}). \tag{6.1}$$

Now ψ also satisfies the relation

$$\psi(\xi, \eta) = \frac{R^2}{2\pi}\int_0^{2\pi}\left(\frac{1}{r_1^2}-\frac{1}{r_2^2}\right)\psi[R, \theta]\, d\theta - \frac{1}{2\pi}\iint_{r<R}\left(\log\frac{r_1 R}{r_2 r_3}\right)(q-\lambda)\psi\, r dr d\theta, \tag{6.2}$$

where (ξ, η) is inside the circle, r_1 is its distance from (x, y), r_2 the distance of the inverse point from (x, y), and r_3 the distance of (ξ, η) from the centre. This is easily proved by applying the formula

$$\iint (u\nabla^2 v - v\nabla^2 u)\, dxdy = \int\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds, \tag{6.3}$$

where $\partial/\partial n$ denotes differentiation along the outward normal, with $u = \psi$ and $v = \log(r_1 R/r_2 r_3)$. Take $\xi = 0$, $\eta = R\epsilon$. Then

$$r_3 = R(1-\epsilon),$$

and, if $x = R\sin\theta$, $y = R(1-\cos\theta)$,

$$r_1^2 = R^2\{2(1-\epsilon)(1-\cos\theta)+\epsilon^2\}.$$

Also $r_2 r_3 = r_1 R$. Hence

$$\frac{1}{r_1^2}-\frac{1}{r_2^2} = \frac{1}{r_1^2}\{1-(1-\epsilon)^2\} = \frac{2\epsilon-\epsilon^2}{R^2\{2(1-\epsilon)(1-\cos\theta)+\epsilon^2\}} \sim \frac{2\epsilon}{R^2(\theta^2+\epsilon^2)}$$

for small ϵ and θ. Hence the first term on the right of (6.2) is

$$O\left\{\int_0^\pi \frac{\epsilon}{\theta^2+\epsilon^2}|\psi|\,d\theta\right\} = O\left\{\frac{1}{\epsilon}\int_0^\epsilon |\psi|\,d\theta\right\}+O\left\{\epsilon\int_\epsilon^\pi |\psi|\frac{d\theta}{\theta^2}\right\}$$

$$= O\left\{\frac{I(\epsilon)}{\epsilon}\right\}+O\left\{\epsilon\left[\frac{I(\theta)}{\theta^2}\right]_\epsilon^\pi+2\epsilon\int_\epsilon^\pi \frac{I(\theta)}{\theta^3}\,d\theta\right\} = O(\epsilon^{\frac{1}{2}}),$$

by (6.1).

Again, (6.3) with

$$u = 1-r^2/R^2, \qquad v = \log(r_1 R/r_2 r_3),$$

gives

$$\iint_{r<R} \log\frac{r_2 r_3}{r_1 R}\,r\,dr\,d\theta = \tfrac{1}{2}\pi(R^2-r_3^2).$$

Thus the second term on the right of (6.2) is

$$O\left\{\iint_{r<R} \log\frac{r_2 r_3}{r_1 R}\,r\,dr\,d\theta\right\} = O(R^2-r_3^2) = O(\epsilon).$$

Hence

$$\psi(\xi,\eta) = O(\epsilon^{\frac{1}{2}}). \tag{6.4}$$

Again, the formula obtained by differentiating (3.4) with respect to y_0 is (taking now $x = R\cos\theta$, $y = R\sin\theta$)

$$\frac{\partial\psi_n}{\partial y_0} = \frac{1}{\pi R}\int_0^{2\pi} \psi_n(x_0+R\cos\theta,\, y_0+R\sin\theta)\sin\theta\,d\theta-$$

$$-\frac{1}{2\pi}\iint_{r<R}\left(\frac{1}{r}-\frac{r}{R^2}\right)(q-\lambda)\psi_n \sin\theta\,r\,dr\,d\theta.$$

Taking (x_0, y_0) at distance ϵ from the boundary of D, and $R = \frac{1}{2}\epsilon$, it follows from (6.4) that

$$\frac{\partial\psi}{\partial n} = O(\epsilon^{-\frac{1}{2}}) \tag{6.5}$$

uniformly except possibly in the neighbourhood of the angles, where we only get

$$\frac{\partial\psi}{\partial n} = O(\epsilon^{-1}). \tag{6.6}$$

7. The Parseval formula

Let $f(x,y)$ be L^2 over D, and let

$$c_n = \iint_D f(x,y)\psi_n(x,y)\,dxdy.$$

Then

$$\sum_{n=0}^{\infty} c_n^2 = \iint_D \{f(x,y)\}^2\,dxdy.$$

As in the case of one variable, it is sufficient to prove this for a special class of functions, e.g. such that f has continuous partial derivatives up to the second order throughout the b-square, and is zero outside D. Let $\psi_n^{(\nu)}$, $c_n^{(\nu)}$ be the eigenfunctions and Fourier coefficients of f for the b-square and $q^{(\nu)}$. Then it is known that

$$\sum_{n=0}^{\infty} c_n^{(\nu)2} = \int_{-b}^{b}\int_{-b}^{b} f^2\,dxdy = \iint_D f^2\,dxdy.$$

Also $c_n^{(\nu)} \to c_n$ for each n. Hence it is sufficient to prove that the series on the left converges uniformly with respect to ν. Now

$$\int_{-b}^{b}\int_{-b}^{b} (f\nabla^2\psi_n^{(\nu)} - \psi_n^{(\nu)}\nabla^2 f)\,dxdy = 0,$$

i.e.

$$\int_{-b}^{b}\int_{-b}^{b} \{f(q^{(\nu)} - \lambda_n^{(\nu)})\psi_n^{(\nu)} - \psi_n^{(\nu)}\nabla^2 f\}\,dxdy = 0,$$

i.e.

$$c_n^{(\nu)} = c_n^{(\nu)*}/\lambda_n^{(\nu)},$$

where $c_n^{(\nu)*}$ is the 'Fourier coefficient' of $f^* = qf - \nabla^2 f$. Also

$$|c_n^{(\nu)*}| \leqslant \left\{\int_{-b}^{b}\int_{-b}^{b} f^{*2}\,dxdy \int_{-b}^{b}\int_{-b}^{b} \psi_n^{(\nu)2}\,dxdy\right\}^{\frac{1}{2}} = \left\{\iint_D f^{*2}\,dxdy\right\}^{\frac{1}{2}},$$

$$\frac{1}{\lambda_n^{(\nu)}} \leqslant \frac{1}{\lambda_n^{(1)}},$$

and

$$\sum \frac{1}{(\lambda_n^{(1)})^2}$$

is convergent. The result therefore follows.

8. The Green's function

Let $G^{(\nu)}(x,y,\xi,\eta,\lambda)$ be the Green's function for the b-square and $q^{(\nu)}$, and let

$$g(x,y,\xi,\eta) = \begin{cases} \dfrac{1}{2\pi}\log\dfrac{R}{\rho} - \dfrac{1}{4\pi}\left(1 - \dfrac{\rho^2}{R^2}\right) & (0 \leqslant \rho \leqslant R), \\ 0 & (\rho > R), \end{cases}$$

where $\rho = \sqrt{\{(x-\xi)^2+(y-\eta)^2\}}$.

We apply (6.3), with

$$u = G^{(\nu)}(x, y, \xi, \eta, \lambda), \qquad v = g(x, y, x_0, y_0),$$

to the circle $r \leqslant R$, where

$$r = \sqrt{\{(x-x_0)^2+(y-y_0)^2\}},$$

with small circles round the points (ξ, η) and (x_0, y_0) cut out of it (supposing that the distance between these points is less than R). We have

$$\begin{aligned}\iint \{g(x, y, x_0, y_0)\nabla^2 G^{(\nu)}(x, y, \xi, \eta, \lambda) - \\ - G^{(\nu)}(x, y, \xi, \eta, \lambda)\nabla^2 g(x, y, x_0, y_0)\}\, dxdy \\ = \iint \Big[g(x, y, x_0, y_0)\{q(x, y)-\lambda\}G^{(\nu)}(x, y, \xi, \eta, \lambda) - \\ - \frac{1}{\pi R^2} G^{(\nu)}(x, y, \xi, \eta, \lambda)\Big]\, dxdy.\end{aligned}$$

The boundary $r = R$ contributes zero, since g and $\partial g/\partial n$ vanish round it. The small circle round (ξ, η) contributes $g(\xi, \eta, x_0, y_0)$, and that round (x_0, y_0) contributes $G^{(\nu)}(x_0, y_0, \xi, \eta, \lambda)$. The result can be written as

$$\begin{aligned}G^{(\nu)}(x_0, y_0, \xi, \eta, \lambda) - g(x_0, y_0, \xi, \eta) = \frac{1}{\pi R^2}\iint_{r<R} G^{(\nu)}(x, y, \xi, \eta, \lambda)\, dxdy + \\ + \frac{1}{2\pi}\iint_{r<R}\left\{\frac{1}{2}\left(1-\frac{r^2}{R^2}\right)-\log\frac{R}{r}\right\}\{q(x, y)-\lambda\}G^{(\nu)}(x, y, \xi, \eta, \lambda)\, dxdy.\end{aligned}$$

The result is the same if the distance between (ξ, η) and (x_0, y_0) is greater than R, since then $g(x_0, y_0, \xi, \eta) = 0$.

It follows that, for given x_0, y_0, R, and λ,

$$\begin{aligned}\int_{-b}^{b}\int_{-b}^{b} |G^{(\nu)}(x_0, y_0, \xi, \eta, \lambda)|^2\, d\xi d\eta \\ = O\Big\{\int_{-b}^{b}\int_{-b}^{b}\int_{-b}^{b}\int_{-b}^{b} |G^{(\nu)}|^2\, dxdyd\xi d\eta\Big\} + O(1) \\ = O\Big\{\sum_{n=0}^{\infty}\frac{1}{|\lambda_n^{(\nu)}-\lambda|^2}\Big\} + O(1) = O(1).\end{aligned}$$

Hence, as $\nu \to \infty$,

$$\sum_{n=0}^{\infty}\frac{\{\psi_n^{(\nu)}(x_0, y_0)\}^2}{|\lambda_n^{(\nu)}-\lambda|^2} = O(1).$$

Accordingly
$$\sum_{n=0}^{\infty} \frac{\{\psi_n(x_0, y_0)\}^2}{|\lambda_n - \lambda|^2} \tag{8.1}$$
is convergent. Hence, for given x_0, y_0, and λ,
$$G_N(x_0, y_0, \xi, \eta, \lambda) = \sum_{n=0}^{N} \frac{\psi_n(x_0, y_0)\psi_n(\xi, \eta)}{\lambda_n - \lambda}$$
converges in mean square over D, say to $G(x_0, y_0, \xi, \eta, \lambda)$.

Now
$$\iint_{r<R} \{G_N(x, y, \xi, \eta, \lambda)\nabla^2 g - g\nabla^2 G_N\}\, dxdy = G_N(x_0, y_0, \xi, \eta, \lambda),$$
and
$$\nabla^2 g = 1/(\pi R^2),$$
$$\nabla^2 G_N = \sum_{n=0}^{N} \frac{\psi_n(\xi, \eta)\{q(x, y) - \lambda_n\}\psi_n(x, y)}{\lambda_n - \lambda}$$
$$= \{q(x, y) - \lambda\}G_N - \sum_{n=0}^{N} \psi_n(x, y)\psi_n(\xi, \eta).$$
Hence
$$G_N(x_0, y_0, \xi, \eta, \lambda) = \frac{1}{\pi R^2} \iint_{r<R} G_N(x, y, \xi, \eta, \lambda)\, dxdy +$$
$$+ \frac{1}{2\pi} \iint_{r<R} \left\{\frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log\frac{R}{r}\right\}\{q(x, y) - \lambda\}G_N(x, y, \xi, \eta, \lambda)\, dxdy -$$
$$- \frac{1}{2\pi} \iint_{r<R} \left\{\frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log\frac{R}{r}\right\} \sum_{n=0}^{N} \psi_n(x, y)\psi_n(\xi, \eta)\, dxdy.$$
Now multiply by any $f(\xi, \eta)$ of L^2, integrate over D, and make $N \to \infty$. We obtain
$$\iint_D G(x_0, y_0, \xi, \eta, \lambda) f(\xi, \eta)\, d\xi d\eta$$
$$= \frac{1}{\pi R^2} \iint_D f(\xi, \eta)\, d\xi d\eta \iint_{r<R} G(x, y, \xi, \eta, \lambda)\, dxdy +$$
$$+ \frac{1}{2\pi} \iint_D f(\xi, \eta)\, d\xi d\eta \iint_{r<R} \left\{\frac{1}{2}\left(1 - \frac{r^2}{R^2}\right) - \log\frac{R}{r}\right\} \times$$
$$\times \{q(x, y) - \lambda\}G(x, y, \xi, \eta, \lambda)\, dxdy +$$
$$+ \iint_D f(\xi, \eta) g(x_0, y_0, \xi, \eta)\, d\xi d\eta, \tag{8.2}$$

the last term by the Parseval theorem. This is of the form

$$\iint_D f(\xi,\eta)F(\xi,\eta)\,d\xi d\eta = 0,$$

where f is arbitrary. It follows that $F = 0$ almost everywhere, i.e. that

$$G(x_0,y_0,\xi,\eta,\lambda)-g(x_0,y_0,\xi,\eta) = \frac{1}{\pi R^2}\iint_{r<R} G(x,y,\xi,\eta,\lambda)\,dxdy+$$

$$+\frac{1}{2\pi}\iint_{r<R}\left\{\frac{1}{2}\left(1-\frac{r^2}{R^2}\right)-\log\frac{R}{r}\right\}\{q(x,y)-\lambda\}G(x,y,\xi,\eta,\lambda)\,dxdy \quad (8.3)$$

almost everywhere.

Now the right-hand side is a continuous function of ξ and η; for the above argument shows that (8.1) converges uniformly with respect to x_0 and y_0 over any region interior to D. Since each $\psi_n(x_0,y_0)$ is continuous, it follows that

$$\sum_{n=0}^{\infty}\frac{\{\psi_n(x_0,y_0)-\psi_n(x_0',y_0')\}^2}{|\lambda_n-\lambda|^2}\to 0$$

as $(x_0',y_0')\to(x_0,y_0)$. Now

$$\frac{1}{R^2}\iint_{r<R} G(x,y,\xi,\eta,\lambda)\,dxdy = \sum_{n=0}^{\infty}\frac{d_n\psi_n(\xi,\eta)}{\lambda_n-\lambda},$$

where the d_n are the 'Fourier coefficients' of

$$h(x,y) = \begin{cases} 1/R^2 & (r\leqslant R),\\ 0 & (r>R).\end{cases}$$

Hence

$$\left|\frac{1}{R^2}\iint_{r<R}\{G(x,y,\xi,\eta,\lambda)-G(x,y,\xi',\eta',\lambda)\}\,dxdy\right|$$

$$\leqslant\left\{\sum_{n=0}^{\infty}d_n^2\sum_{n=0}^{\infty}\frac{\{\psi_n(\xi,\eta)-\psi_n(\xi',\eta')\}^2}{|\lambda_n-\lambda|^2}\right\}^{\frac{1}{2}}\to 0$$

as $(\xi',\eta')\to(\xi,\eta)$. Thus the first term on the right of (8.3) is continuous, and, similarly, so is the second term. Hence the left-hand side is almost everywhere equal to a continuous function, and so may be taken to be equal to it everywhere, since it is so far defined only almost everywhere.

In particular it follows that, as $(\xi,\eta)\to(x,y)$,

$$G(x,y,\xi,\eta,\lambda) = \frac{1}{2\pi}\log\frac{1}{\rho}+O(1). \quad (8.4)$$

We can also differentiate (8.3) with respect to x_0 or y_0, and from the differentiated formula we obtain

$$\frac{\partial G}{\partial \rho} = -\frac{1}{2\pi\rho} + O(1). \tag{8.5}$$

Finally, on making $R \to 0$ we obtain

$$\nabla^2 G + \{\lambda - q(x,y)\} G = 0, \tag{8.6}$$

except when $(x,y) = (\xi,\eta)$.

9. The convergence theorem

Let $f(x,y)$ have bounded first-order partial derivatives in D, let $\nabla^2 f$ be L^2 over D, and let f vanish round the boundary, in the sense that $f = O(\epsilon)$ within a distance ϵ of the boundary.

We shall now assume that we can approximate to the boundary of D by curves C of bounded length, having continuously turning tangents. Then

$$\iint_{D'} (f\nabla^2\psi_m - \psi_m \nabla^2 f)\, d\xi d\eta = \int_C \left(f\frac{\partial \psi_m}{\partial n} - \psi_m \frac{\partial f}{\partial n}\right) ds,$$

where D' is the region enclosed by C, and $\partial/\partial n$ denotes differentiation along the normal to C. If C is within ϵ of the boundary,

$$f = O(\epsilon), \qquad \frac{\partial \psi_m}{\partial n} = o\left(\frac{1}{\epsilon}\right), \qquad \psi_m = o(1), \qquad \frac{\partial f}{\partial n} = O(1).$$

Hence the right-hand side tends to 0, and so

$$\iint_D (f\nabla^2\psi_m - \psi_m \nabla^2 f)\, d\xi d\eta = 0,$$

i.e.

$$\iint_D \{f(q-\lambda_m)\psi_m - \psi_m \nabla^2 f\}\, d\xi d\eta = 0.$$

Hence

$$c_m = \frac{1}{\lambda_m - \lambda} \iint_D \psi_m (q - \nabla^2 - \lambda) f\, d\xi d\eta \tag{9.1}$$

if λ is not equal to any of the λ_m.

If (x_0, y_0) is an interior point of D, and $h(x,y)$, d_n are defined as above, then

$$\frac{1}{R^2} \iint_{r<R} f(x,y)\, dxdy = \iint_D f(x,y) h(x,y)\, dxdy$$

$$= \sum_{n=0}^{\infty} c_n d_n$$

by the Parseval theorem. Now, by (9.1),

$$\sum_{n=0}^{N} c_n d_n$$

$$= \sum_{n=0}^{N} \frac{1}{\lambda_n - \lambda} \iint_D \iint_D h(x,y)\psi_n(x,y)\psi_n(\xi,\eta)(q-\nabla^2-\lambda)f(\xi,\eta)\,dxdyd\xi d\eta$$

$$= \iint_D \iint_D h(x,y)G_N(x,y,\xi,\eta,\lambda)(q-\nabla^2-\lambda)f(\xi,\eta)\,dxdyd\xi d\eta$$

$$\to \iint_D \iint_D h(x,y)G(x,y,\xi,\eta,\lambda)(q-\nabla^2-\lambda)f(\xi,\eta)\,dxdyd\xi d\eta.$$

Hence

$$\frac{1}{R^2} \iint_{r<R} \left\{ f(x,y) - \iint_D G(x,y,\xi,\eta,\lambda)(q-\nabla^2-\lambda)f(\xi,\eta)\,d\xi d\eta \right\} dxdy = 0.$$

By the analysis of the previous section, the (x,y)-integrand is continuous. Hence, making $R \to 0$,

$$f(x,y) = \iint_D G(x,y,\xi,\eta,\lambda)(q-\nabla^2-\lambda)f(\xi,\eta)\,d\xi d\eta$$

$$= \lim_{N\to\infty} \iint_D G_N(x,y,\xi,\eta,\lambda)(q-\nabla^2-\lambda)f(\xi,\eta)\,d\xi d\eta$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} \frac{\psi_n(x,y)}{\lambda_n-\lambda} \iint_D \psi_n(\xi,\eta)(q-\nabla^2-\lambda)f(\xi,\eta)\,d\xi d\eta$$

$$= \lim_{N\to\infty} \sum_{n=0}^{N} c_n \psi_n(x,y).$$

Hence under the above conditions

$$f(x,y) = \sum_{n=0}^{\infty} c_n \psi_n(x,y).$$

10. We can also prove that the solution of the problem

$$\{\nabla^2+(\lambda-q)\}\Phi = f \quad \text{in } D, \tag{10.1}$$

$\Phi = 0$ round the boundary of D, is

$$\Phi(x,y,\lambda) = -\iint_D G(x,y,\xi,\eta,\lambda)f(\xi,\eta)\,d\xi d\eta, \tag{10.2}$$

at any rate provided that $f^* = qf - \nabla^2 f$ is L^2 over D. For (8.2) is equivalent to

$$\Phi(x,y,\lambda) = \frac{1}{\pi R^2}\iint_{r<R} \Phi(\xi,\eta,\lambda)\,d\xi d\eta + \\ + \frac{1}{2\pi}\iint_{r<R}\left\{\frac{1}{2}\left(1-\frac{r^2}{R^2}\right)-\log\frac{R}{r}\right\}\{(q-\lambda)\Phi+f\}\,d\xi d\eta,$$

whence (10.1) follows on making $R \to 0$. Also (10.2) is equivalent to

$$\Phi(x,y,\lambda) = \sum_{n=0}^{\infty} \frac{c_n \psi_n(x,y)}{\lambda-\lambda_n}.$$

If f^* is L^2, $c_n = c_n^*/\lambda_n$, where $\sum c_n^{*2} < \infty$. Hence

$$\Phi(x,y,\lambda) = \sum_{n=0}^{\infty} c_n^* \frac{\psi_n(x,y)}{\lambda_n(\lambda-\lambda_n)},$$

and this is uniformly convergent, since $\psi_n = O(\lambda_n^{\frac{1}{2}})$. Since each $\psi_n \to 0$ at the boundary, so does Φ.

NOTE ON A THEOREM OF HARDY AND ROGOSINSKI

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Introduction

HARDY AND ROGOSINSKI [1] have proved a series of theorems on functions subharmonic (s.h.) in a strip, of which the principal is Theorem 2 below on the convexity of $\bar{\phi}(x) = \overline{\lim_{y\to\infty}} f(x,y)$. In their proof, a 'delicate argument' was required to prove $\bar{\phi}(x)$ upper semi-continuous (u.s.c.). In this note a much simpler proof is obtained by using well-known Phragmén–Lindelöf methods. In fact, I use the maximum principle for s.h. functions in the following slightly generalized form.

THEOREM 1. *If* (i) $f(x,y)$ *is s.h. in the open half-strip* D

$$(\alpha < x < \beta,\ y > \gamma),$$

(ii) $f(x,y)$ *is u.s.c. and bounded above in the closure* $\bar{D}$ *of* D,*

then the maximum principle is valid in D *according to the boundary* R *of* D, *i.e.*

$$\text{l.u.b.}_{(x,y)\in D} f(x,y) \leqslant \text{l.u.b.}_{(x,y)\in R} f(x,y). \tag{1.1}$$

Theorem 1 is known (see, for example, [2]), but for the sake of completeness, I outline its simple proof. Let

$$f_\epsilon(x,y) = f(x,y) + \epsilon(x^2 - y^2) = f + \epsilon g \quad (\epsilon > 0), \tag{1.2}$$

let D_Y denote the sub-domain of D

$$\alpha < x < \beta, \qquad \gamma < y < Y \tag{1.3}$$

and let R_Y denote the bottom and sides of this rectangle, i.e. the part of the boundary of D_Y that is contained in R. Since f is bounded above and $g \to -\infty$ uniformly in (α, β) as $y \to \infty$, it follows that for a fixed ϵ we can choose Y so that

$$\text{l.u.b.}_{\alpha<x<\beta} f_\epsilon(x,Y) < \text{l.u.b.}_{(x,y)\in R_Y} f_\epsilon(x,y), \tag{1.4}$$

* Strictly, $\bar{D}$ is the finite part of the closure of D, and R is the finite part of its boundary.

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and, g being harmonic, it follows by the usual maximum principle for functions s.h. in a bounded domain [3] that (1.1) is true for f_ϵ in D_Y according to R_Y and hence according to R. Y being as large as we please, for every (x, y) of D

$$f(x,y)+\epsilon g(x,y) \leqslant \underset{(x,y)\in R}{\text{l.u.b.}}\, f_\epsilon(x,y) \leqslant \underset{(x,y)\in R}{\text{l.u.b.}}\, f(x,y)+\epsilon \max(\alpha^2,\beta^2) \quad (1.5)$$

and, by letting $\epsilon \to 0$, Theorem 1 follows.

THEOREM 2. *If $f(x, y)$ is s.h. and bounded above in the open half-strip $\alpha < x < \beta$, $y > \gamma$, then*

either (a) $\bar{\phi}(x) = \overline{\lim_{y\to\infty}} f(x,y) = -\infty$ *for* $\alpha < x < \beta$,

or (b) $\bar{\phi}(x)$ *is continuous and convex for* $\alpha < x < \beta$.

Let x_1, x_2 satisfy $\alpha < x_1 < x_2 < \beta$. I first prove that

$$\bar{\phi}(x) \leqslant \max\{\bar{\phi}(x_1), \bar{\phi}(x_2)\} \quad \text{for} \quad x_1 < x < x_2. \tag{2.1}$$

Suppose first that x_1 is positive and that

$$\max\{\bar{\phi}(x_1), \bar{\phi}(x_2)\} = \mu > -\infty. \tag{2.2}$$

Then given $\delta > 0$, we can choose η such that

$$f(x_1,y) < \mu+\delta, \quad f(x_2,y) < \mu+\delta, \quad \text{for} \quad y \geqslant \eta. \tag{2.3}$$

Now let $$f_n(x,y) = f(x,y)-nh(x,y),$$

where $$h(x,y) = x/\{x^2+(y-\eta)^2\}. \tag{2.4}$$

Then h is harmonic and $\lim_{y\to\infty} h(x,y) = 0$. Also we can choose n so that

$$f_n(x,\eta) = f-n/x < \mu \quad \text{for} \quad x_1 \leqslant x \leqslant x_2. \tag{2.5}$$

Since $f_n(x, y)$ satisfies the conditions of Theorem 1 in the half-strip $x_1 \leqslant x \leqslant x_2$, $y \geqslant \eta$, it follows that for all (x, y) in this half-strip

$$f_n(x,y) < \mu+\delta. \tag{2.6}$$

Hence

$$\bar{\phi}(x) = \overline{\lim_{y\to\infty}} f(x,y) = \overline{\lim_{y\to\infty}} f_n(x,y) \leqslant \mu+\delta \quad \text{for} \quad x_1 < x < x_2. \tag{2.7}$$

Since δ is arbitrary, (2.1) follows. Trivial modifications are required if $\mu = -\infty$ or $x_1 \leqslant 0$.

To complete the proof of the theorem, I use the argument familiar in the proof of the '3 lines theorem'. For arbitrary real λ, $f(x, y)-\lambda x$ satisfies the conditions of the theorem. Hence, by (2.1),

$$\bar{\phi}(x)-\lambda x \leqslant \max\{\bar{\phi}(x_1)-\lambda x_1, \bar{\phi}(x_2)-\lambda x_2\} \quad \text{for} \quad x_1 < x < x_2. \tag{2.8}$$

If $\bar{\phi}(x_1) > -\infty$ and $\bar{\phi}(x_2) > -\infty$, we can choose λ so that

$$\bar{\phi}(x_1)-\lambda x_1 = \bar{\phi}(x_2)-\lambda x_2 \tag{2.9}$$

and the convexity of $\bar{\phi}(x)$ follows. Trivial modifications are required if $\bar{\phi}(x_1) = -\infty$ or $\bar{\phi}(x_2) = -\infty$. Finally, it is well known that a convex function, bounded above, is either continuous or identically $-\infty$.

REFERENCES

1. G. H. Hardy and W. W. Rogosinski, *Proc. Royal Soc.* A, 185 (1946), 1–14.
2. G. O. Thorin, *Convexity Theorems.* Lund Univ. Mat. Semin. 9 (1948), 15.
3. J. E. Littlewood, *Lectures on the Theory of Functions* (Oxford, 1944), 156.

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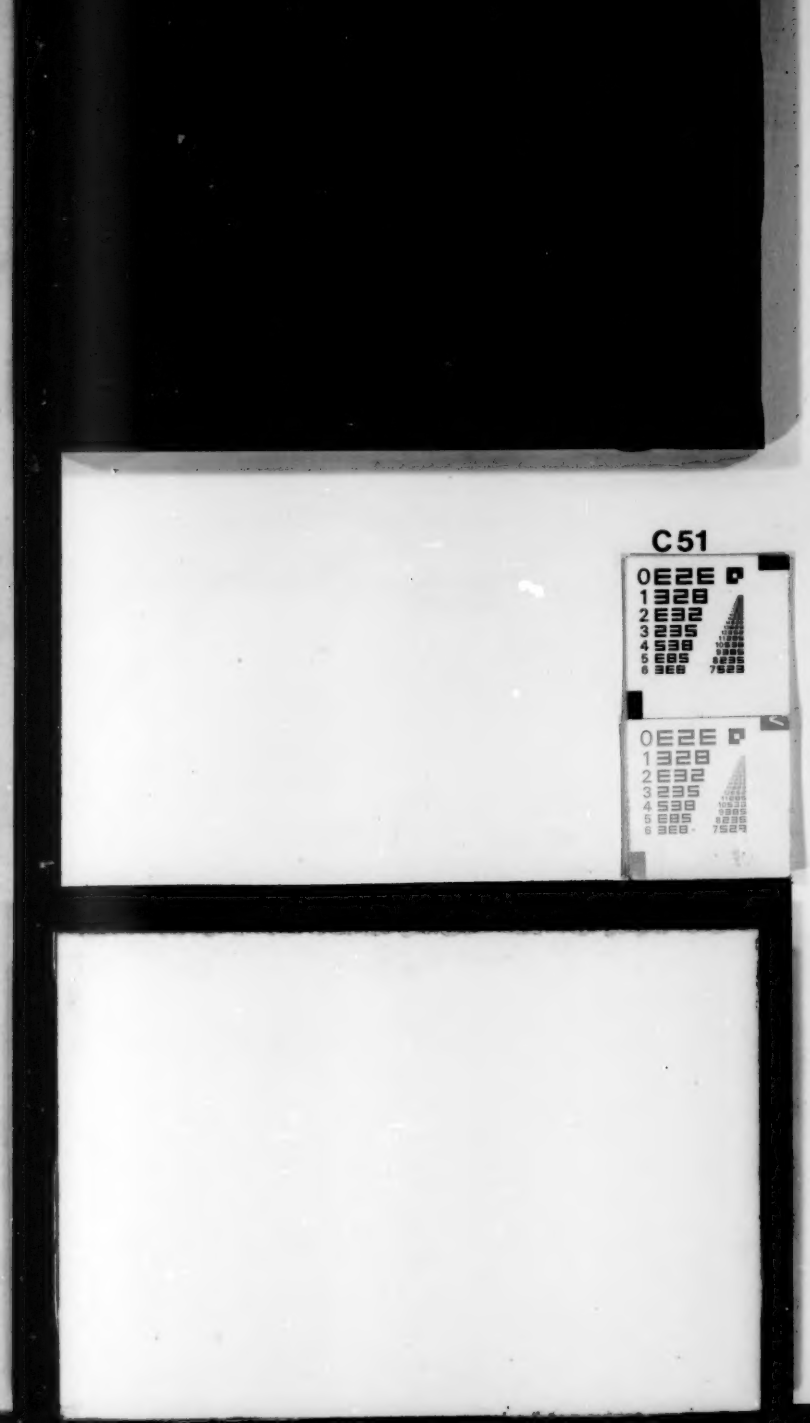
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